ON THE MINIMAL MODULES FOR EXCEPTIONAL LIE ALGEBRAS: JORDAN BLOCKS AND STABILISERS

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Abstract. Let G be a simple simply-connected exceptional algebraic group of type G_2, F_4, E_6 or E_7 over an algebraically closed field k of characteristic $p > 0$ with $\mathfrak{g} = \mathrm{Lie}(G)$. For each nilpotent orbit $G \cdot e$ of $\mathfrak{g}$, we list the Jordan blocks of the action of e on the minimal induced module $V_{\min}$ of $\mathfrak{g}$. We also establish when the centralisers G_v of vectors $v \in V_{\min}$ and stabilisers $\mathrm{Stab}_G\langle v\rangle$ of 1-spaces $\langle v\rangle \subset V_{\min}$ are smooth; that is, when $\dim G_v = \dim \mathfrak{g}_v$ or $\dim \mathrm{Stab}_G\langle v\rangle = \dim \mathrm{Stab}_{\mathfrak{g}}\langle v\rangle$.

1. Introduction

Let G be a simply-connected exceptional algebraic group over an algebraically closed field k of characteristic $p \geq 0$ with $\mathfrak{g} = \mathrm{Lie}(G)$. It is a basic fact of the theory of algebraic groups that G is defined over $\mathbb{Z}$; that is, there is a group $G_{\mathbb{Z}}$ such that after extension of scalars to k, one gets the group G. If G is not of type E_8, then $G_{\mathbb{Z}}$ admits a non-trivial module $(V_{\min})_{\mathbb{Z}}$ of smaller dimension. After reduction modulo p, one then gets a module $V_{\min}$ for G. In case $G = G_2, F_4, E_6$ or E_7 such a module has dimension $7, 26, 27$ or 56, respectively. Recall also that G acts via the adjoint action on its Lie algebra; the associated representation is called the adjoint module. With a classification of unipotent elements in hand, the Jordan block sizes of the action of unipotent elements of G on the adjoint module $\mathfrak{g}$ and minimal module $V_{\min}$ were computed in [Law95] (see also [Law98]) and have been used extensively by the mathematical community. Recall that the characteristic p is good for the exceptional group G if $p > 3$ and if G is of type E_8, $p > 5$. In good characteristic one has a Springer morphism: a G-equivariant bijective map between the variety of unipotent elements of G and the nilpotent cone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$. Thus the classification of orbits of nilpotent and unipotent elements is the same. At the beginning of [UGA05, §3], a reference to a private communication with Lawther indicates that he has checked that the Jordan blocks of unipotent elements and associated nilpotent elements on the adjoint and minimal modules are always the same in good characteristic, with a single exception: on the minimal 56-dimensional module for E_7 when $p = 5$, the regular nilpotent element has blocks of size $23^2, 10$ whereas the regular unipotent element has blocks $24, 22, 10$. A consequence of these calculations is that together with the remaining Jordan block sizes for nilpotent elements in bad characteristic found in [UGA05], the block sizes on the adjoint module are therefore known in all characteristics. One aim of this note is to compute the Jordan block sizes of nilpotent elements on $V_{\min}$, which are new in bad characteristic. For completeness and ease of use, we have included the block sizes in good characteristic also.

Theorem 1.1. *The Jordan blocks of nilpotent elements e on $V_{\min}$ are listed in Tables 2, 3 and 4.*

As explained above, comparison with the tables in [Law95] yields the following:

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Corollary 1.2 (Lawther)**.** *In good characteristic, the Jordan block sizes on $V_{\min}$ of nilpotent and unipotent elements of the same label are the same, unless $p = 5$ and $G = E_7$, where only the Jordan block sizes of the regular unipotent and nilpotent elements disagree.*

Using the calculations in [Law95], one sees by inspection that the number of Jordan blocks of unipotent elements on the adjoint module is independent of good characteristic; this is reflecting the fact that the centralisers of unipotent elements G_u are smooth. Another way of stating this is that the orbit $G \cdot u$ of u is separable, or that $\mathrm{Lie}(C_G(u)(k)) = \mathfrak{c}_{\mathfrak{g}}(u)$. The phenomenon that centralisers are usually smooth holds much more generally; see [BMRT10] and more recently, [Her13]. It was also noted in [Law95] when the number of Jordan blocks of a unipotent element on $V_{\min}$ was the same as in characteristic zero. (It turned out that this held in good characteristic.) Thus in good characteristic the scheme of fixed points $(V_{\min})^u$ is smooth.

We discuss the complementary question in our context, which is possibly more natural. Beforehand we must first be a little more precise about $V_{\min}$. Most of the time $V_{\min}$ is irreducible and the theory of high weights identifies $V_{\min}$ uniquely up to isomorphism (possibly after twisting with a graph automorphism in the case of E_6). The two exceptions are when $(G, p) = (F_4, 3)$ or $(G_2, 2)$. In this case there are essentially two ways in which one may construct a lattice in $(V_{\min})_{\mathbb{Z}}$. For one of these, the resulting module after reduction modulo p, henceforth $V_{\min}$, has a 1-dimensional trivial module in its head with an irreducible $(\dim V_{\min} - 1)$-dimensional socle; this is an *induced*, *co-standard* or *dual-Weyl* module for G, and $V_{\min}^*$ is the corresponding *standard* or *Weyl* module for G. (Note that for the purposes of computing ranks of powers of matrices, hence Jordan blocks, it matters not whether one works with a module V or its dual.)

With this clarification in hand, let $v \in V_{\min}$ and $\langle v \rangle = kv$ be the 1-space it spans over k. Then we establish when the stabilisers G_v and $\mathrm{Stab}_G\langle v \rangle$ are smooth; by [Jan03, I.7.18(5)] this occurs precisely when $\mathfrak{g}_v = \mathrm{Lie}(G_v(k))$ and $\mathrm{Stab}_{\mathfrak{g}}\langle v \rangle = \mathrm{Lie}(\mathrm{Stab}_G\langle v \rangle(k))$, respectively. This amounts again to establishing when the orbit $G \cdot v$ (or $G \cdot \langle v \rangle$) is separable.

One consequence is the following:

Theorem 1.3. *The stabilisers G_v of vectors $v \in V_{\min}$ and $\mathrm{Stab}_G\langle v \rangle$ of 1-spaces of $V_{\min}$ are smooth whenever p is a good prime for G.*

We were unable to find any general results in this direction. Thus we ask the following (probably too general) question:

Question 1.4. *Let G be a reductive algebraic group over an algebraically closed field k and V be a restricted G-module. Under what circumstances are the centralisers G_U or stabilisers $\mathrm{Stab}_G(U)$ of subschemes $U \subseteq V$ smooth algebraic groups?*

Let us reiterate that the paper [Her13] gives an answer to the question about centralisers when V is the adjoint module for G; the correct condition on the characteristic p of k is that it be *pretty good* for G. Also, Cartier's theorem tells us that all algebraic groups are smooth over fields of characteristic zero, so there should be some hope that there are answers which involve a bound on p. (Possibly $p > \dim V$ might suffice for the answer to be yes.) Note that V must be restricted for the answer to the question to be interesting: if V is a Frobenius twist $W^{[1]}$ of another non-trivial G-module W, then $\mathfrak{g}$ acts trivially on V, hence centralises every vector, but G certainly does not. Futhermore, let us underline that already the question is interesting for the case that U consists of a single k-point of V. The part of the question dealing with stabilisers is much less likely to have a nice answer: see [HS16] which basically gives the answer for the corresponding question about normalisers of subspaces on the adjoint module.

1.1. **Comparison with UGA VIGRE results:** During the process of making these computations, it was discovered that there are a number of errors in the tables of representatives of nilpotent orbits in [UGA05]. In the appendix we have provided a complete list of corrected representatives derived from [LS12] and [Spa84] that can be used to fix the UGA VIGRE tables.

The first set of (minor) errors involve the transcription of representatives from the MAGMA code used into LaTeX, yet the stated Jordan block sizes remain correct. These orbit representatives are

- $F_4(a_3)$ in F_4;
- $A_6^{(2)}$ in E_8;
- $E_7(a_1)$ in E_8;
- $E_8(a_2)$, $E_8(a_3)$, $E_8(a_4)$ in E_8.

With these corrections the UGA VIGRE representatives are correct for $p = 0$ and $p \geq 5$.

The second set of errors orbits involve the following orbits in characteristics 2 or 3:

- $D_r(a_1)$, $D_r(a_1) + A_s$, $p = 2$;
- $D_r(a_2)$, $D_r(a_2) + A_s$, $p = 3$;
- $E_8(b_6)$, $p = 2$ and 3.

The problem is that the aforementioned listed representatives in [UGA05] break down in either characteristic 2 or 3, meaning that the Jordan block sizes are not correct in these cases. One can use the tables in the appendix for these orbits and their Jordan blocks to correct the tables in [UGA05]. In fact, we give all adjoint Jordan block sizes in all cases for completeness, even where they are known to coincide with those in [Law95] for good characteristic.

2. Jordan blocks

We describe the method of computation of the tables of Jordan blocks. All this was done in GAP. For convenience, we locate the relevant modules $V_{\min}$ in subquotients of nilradicals of parabolic subalgebras of $\mathfrak{g}$. The general theory on the structure of nilradicals of parabolic subalgebras of reductive Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ (and the group-theoretic analogues for G) can be found in [ABS90]. With the notation of [Bou82] we will give a description in terms of roots. In $\mathfrak{g}$ of type E_8 with Cartan subalgebra $\mathfrak{h}$, we locate a Levi subalgebra $\mathfrak{l}$ of type E_7 containing $\mathfrak{h}$ and corresponding roots of the form

$$\overset{*****\,*\,0}{*},$$

where each value of $*$ is taken arbitrarily so that the result is a root. Then the derived subalgebra $\mathfrak{l}'$ is simple of type E_7 and acts by derivations on the space spanned by the 56 positive roots of the form

$$\overset{******1}{*}$$

An analysis of the highest weight shows that this is indeed the 56-dimensional module $V_{\min} = V_{56}$. Similarly, one locates E_6 in E_7 correponding to roots $\overset{*****0}{*}$ acting on a 27-dimensional minimal module $V_{\min} = V_{27}$ corresponding to roots $\overset{*****1}{*}$. Nilpotent orbit representatives for E_6 and E_7 in all characteristics in terms of a sum of simple root spaces are available from [LS12] and [Spa84].

Now one realises all these elements in GAP. The package `LieAlgebras` in the standard distribution of GAP4 will construct a simple Lie algebra $\mathfrak{g}$ of rank r and number of positive roots $n = |R^+|$ over the rationals which comes with a 'canonical' Chevalley basis B. This has $\{B[1], \dots, B[r]\}$ being a basis for the simple root spaces, $\{B[1], \dots, B[n]\}$ being a basis for the positive root spaces; for $1 \leq i \leq n$ we have $B[n+i]$ spans the root space corresponding to the negative root to which $B[i]$ corresponds, and $\{B[2n+1], \dots, B[2n+r]\}$ spans a Cartan subalgebra. Further, if $a, b \in \mathfrak{g}$, then the operation $\mathtt{a} * \mathtt{b}$ returns the commutator $[a, b]$ expressed via the $B[i]$. It is a simple matter to write any nilpotent representative e as a sum of a subset of the $B[i]$ and also identify those $B[i]$ which span a basis B_V of $V_{\min}$ as described in the previous paragraph. One may then ask GAP to compute the action of e on $V_{\min}$ as a matrix over the basis B_V by hitting each vector of B_V with e and re-expressing this as a linear combination of elements of B_V. This associates e to a $(\dim V_{\min} \times \dim V_{\min})$-matrix M_e whose entries are integers by virtue of the fact that B was a Chevalley basis.

The Jordan blocks of e are then determined by the ranks of successive powers of M_e. We first run a routine which bounds the number of exceptional primes whose Jordan block structure differs from the generic Jordan block structure as seen over $\mathbb{Q}$. This works simply by taking the union over all primes dividing the elementary divisors of each power of M_e. Then the Jordan block structure is output over $\mathbb{Q}$, together with the Jordan block structure over $\mathbb{F}_p$ for any exceptional p. The result for E_6 and E_7 is then output in the tables below.

The situation for the 7- and 26-dimensional minimal modules for the Lie algebras of type G_2 and F_4 is done similarly, but one must work just a little harder. One has that F_4 is a subalgebra of E_6, such that the E_6-module V_{27} has restriction to F_4 which is $V_{\min} \oplus k$ unless $p = 3$ and $V_{27}|F_4$ is uniserial with successive factors k, $\operatorname{soc} V_{\min}$, and k. (It is in fact a tilting module.) If $p \neq 3$ then the module $V_{\min}$ is irreducible and self-dual, so that for all p there is a quotient isomorphic to $V_{\min}$ of dimension 26. Now F_4 is located in E_6 by sending the simple root vector e_{α_1} to e'_{α_2}, e_{α_2} to e'_{α_4}, e_{α_3} to $e'_{\alpha_3} + e'_{\alpha_5}$ and e_{α_4} to $e'_{\alpha_1} + e'_{\alpha_6}$. The remaining positive root elements of F_4 expresed in terms of those of E_6 can be generated from this. If V_{27} has basis B_V it is straightforward to locate a vector stabilised by F_4 and then form the matrix of the action of each element e on the quotient; this gives a 26×26-matrix M_e. One then repeats the procedure as described above for computing the Jordan blocks.

Finally, a similar procedure locates G_2 in a D_4-Levi subalgebra of E_6 sending e_{α_1} to $e'_{\alpha_2} + e'_{\alpha_3} + e'_{\alpha_5}$ in E_6 and e_{α_2} to e'_{α_4} in E_6. Then the 8-dimensional natural module V_8 for D_4 corresponding to roots $\begin{smallmatrix}0{*}{*}{*}0\\{*}\end{smallmatrix}$ is obtained via the roots $\begin{smallmatrix}1{*}{*}{*}0\\{*}\end{smallmatrix}$. The restriction of V_8 to the G_2-subalgebra contains a trivial submodule such that the quotient is isomorphic to $V_{\min}$ as before.

3. Smoothness of centralisers and stabilisers

In this section we consider simple, simply-connected algebraic groups of type $G := G_2$, F_4, E_6 and E_7 over algebraically closed fields k of arbitrary characteristic acting on their minimal modules $V_{\min} := V_7$, V_{26}, V_{27} and V_{56}, of dimensions 7, 26, 27 and 56 respectively. The stabilisers and centralisers of 1-spaces of $V_{\min}$ for the corresponding finite groups $G_2(\mathbb{F}_q)$, $F_4(\mathbb{F}_q)$, $E_6(\mathbb{F}_q)$ and $E_7(\mathbb{F}_q)$ are well-known to group theorists and we record the extension which gives the reduced part $(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$ of the scheme-theoretic stabilisers $\mathrm{Stab}_G\langle v\rangle$ in G of the 1-space $\langle v\rangle \in V_{\min}$ in the next lemma. (Recall that for an algebraic group K, over an algebraically closed field k, not necessarily smooth, one may associate a smooth algebraic group $K_{\mathrm{red}} \subseteq K$ such that $K_{\mathrm{red}}(k) = K(k)$.) We prove, using computational methods, that $\mathrm{Stab}_G\langle v\rangle$ and G_v are smooth provided p is a good prime, so that for these primes, $(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}} = \mathrm{Stab}_G\langle v\rangle$ and the Lie theoretic stabiliser is recovered as $\mathrm{Lie}((\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}})$.

Lemma 3.1. *(i) If G is of type E_7 then $G(k)$ has four orbits on the 1-spaces of $V := V_{56}$. If $0 \neq \langle v\rangle \subseteq V$ then the stabiliser $(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$ is a closed subgroup of G isomorphic to one of:*

- *(a) an E_6 parabolic subgroup of G;*
- *(b) the semidirect product of an E_6 Levi subgroup of G with an involution inducing a graph automorphism on E_6;*
- *(c) a subgroup of a D_6-parabolic subgroup P of G isomorphic to $B_5T_1R_u(P)$;*
- *(d) a subgroup of an E_6-parabolic P of G equal to the semidirect product $HR_u(P)$, where H is a subgroup of the Levi subgroup L of P of type F_4T_1.*

(ii) If G is of type E_6 then $G(k)$ has three orbits on the 1-spaces of $V := V_{27}$. If $0 \neq \langle v\rangle \subseteq V$ then the stabiliser $(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$ is a closed subgroup of G isomorphic to one of:

- *(a) a D_5-parabolic subgroup P of G;*
- *(b) a subgroup of a D_5-parabolic subgroup P equal to the semidirect product $HR_u(P)$, where H is a subgroup of the Levi subgroup L of P of type B_4T_1;*
- *(c) a subgroup of G of type F_4.*

(iii) If G is of type F_4 then $G(k)$ has infinitely many orbits on the 1-spaces of $V := V_{26}$. If $0 \neq \langle v\rangle \subseteq V$ and $p \neq 3$ then the stabiliser $(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$ is isomorphic to one of

- *(a) B_4 (one orbit);*
- *(b) a B_3-parabolic subgroup P of G (one orbit);*
- *(c) a subgroup of P isomorphic to $G_2T_1 \ltimes k^{14}$ (one orbit);*
- *(d) a subgroup of P isomorphic to $B_3 \ltimes k^7$ (one orbit);*
- *(e) a 28-dimensional subgroup such that the connected component $((\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}})^\circ$ has type D_4. In this case, each orbit contains a unique element in one of the $k \setminus \{0\}$ generic 1-spaces of the 0-weight space of F_4 on V_{26}.*

(iv) If G is of type G_2 then $G(k)$ has two orbits on the set of 1-spaces of $V := V_7$. If $0 \neq \langle v\rangle \subseteq V$ then the stabiliser $(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$ is isomorphic to one of

- *(a) a long A_1-parabolic subgroup of G;*
- *(b) $A_2.2$ for a subsystem subgroup of type A_2 consisting of long roots.*

Proof. Let $H := (\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$. First of all, observe that H, as a subgroup of the $\mathbb{Z}$-defined embedding of G into $\mathrm{GL}(V)$ for $V = V_{27}$ or V_{56} is defined over $\mathbb{Z}$, hence certainly over $\bar{\mathbb{F}}_p$. We have $H_{\bar{\mathbb{F}}_p}(\bar{\mathbb{F}}_p) = \bigcup_{r\geq 0} H_{\bar{\mathbb{F}}_p}(\mathbb{F}_{p^r})$, for instance by intersecting with $\mathrm{GL}(V)(\mathbb{F}_{p^r})$. Since H is smooth, the $\bar{\mathbb{F}}_p$-points of H are dense in H and so we have that the union $\bigcup_{r\geq 0} H_{\bar{\mathbb{F}}_p}(\mathbb{F}_{p^r})$ is dense in H.

Assume we are not in case (iii)(e) or (iv) in characteristic 2. Then the structure as stated, in view of [SS70, I.2.7], follows directly from [CC88, p467 & Table 2] (for E_6 and F_4), [LS87, Lemma 4.3] (for E_7) and [Kle88, Prop. 2.2] (for G_2). In reading those references, note that twisted subgroups such as ${}^2A_2(q)$ occur in the presence of a stabliser $\mathrm{Stab}_G\langle v\rangle$ of the the form $H \cdot \langle\tau\rangle$ where τ is a graph automorphism of H. Then the orbit $G \cdot v$ splits into $|\mathrm{H}^1(F, G_v/G_v^\circ)|$ orbits under $G(\mathbb{F}_{p^r})$, by [SS70, I.2.7].

A little more work is necessary to understand (iii)(e). It is shown in [CC88] that each 1-space not conjugate to any previously considered is conjugate to a subspace of a 'special plane' $\pi = \langle e_1, e_2, e_3\rangle$ (in [CC88] a 'plane' is a plane of $\mathbb{P}(V_{27})$, hence a 3-space of V_{27}). Moreover, all such special planes are conjugate under the action of E_6 and one may choose the e_i to be weight vectors for E_6. Since F_4 is in fact the stabiliser of an element $e = e_1+e_2+e_3$ of π, it is not hard to check that the 0-weight space for a maximal torus F_4 in $V_{27}/\langle e\rangle$ is $\pi/\langle e\rangle$. Then a generic 1-space in $\pi/\langle e\rangle$ is $\langle e_1 + t\cdot e_2\rangle + \langle e\rangle$ for $t \neq 0, 1$. In light of [CC88], the stabilisers in G of all generic 1-spaces are then seen to satisfy the conditions in (iii)(e) as stated.

For (iv) in characteristic 2, note that the stabilisers of the 1-spaces given are both maximal smooth subgroups H_1 and H_2 of G which are $\mathbb{Z}$-defined. After reduction modulo 2, we will therefore have containments $H_1 \subseteq (\mathrm{Stab}_G\langle v_1\rangle)_{\mathrm{red}}$ and $H_2 \subseteq (\mathrm{Stab}_G\langle v_2\rangle)_{\mathrm{red}}$. It cannot be the case that either $\mathrm{Stab}_G\langle v_1\rangle$ or $\mathrm{Stab}_G\langle v_2\rangle$ is the whole of G, since G has no fixed 1-spaces on $V_{\min}$, so the isomorphism types of $(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$ must be as given. To see that the number of orbits is still the same, one counts the number of elements in orbits of 1-spaces for $G_2(q)$. If P is a long root parabolic of $G_2(q)$ it is an easy check that

$$|G_2(q)| \cdot \left(\frac{1}{|P(q)|} + \frac{1}{|A_2(q).2|} + \frac{1}{|{}^2A_2(q).2|}\right) = \frac{q^7-1}{q-1}$$

as required. □

Using GAP we find that the group-theoretic and infinitesimal stabilisers of 1-spaces correspond.

Theorem 3.2. *Let G be simple and simply-connected of type E_7 (resp. E_6, F_4, G_2) and let $V = V_{56}$ (resp. $V = V_{27}$, V_{26}, V_7) be a minimal-dimensional non-trivial induced module for G.*

Then the stabilisers in G of vectors and 1-spaces of V are smooth—that is, $\dim G_v = \dim \mathfrak{g}_v$ and $\dim \mathrm{Stab}_G\langle v\rangle = \dim \mathrm{Stab}_{\mathfrak{g}}\langle v\rangle$—with the following exceptions:

- *(i) If $(G, p) = (E_7, 2)$, then G_v and $\mathrm{Stab}_G\langle v\rangle$ are not smooth if $\langle v\rangle$ has stabiliser of type $F_4T_1 \ltimes k^{26}$.*
- *(ii) If $(G, p) = (E_7, 2)$, then $\mathrm{Stab}_G\langle v\rangle$ is not smooth if $\langle v\rangle$ has stabiliser of type $E_6.2$.*
- *(iii) If $(G, p) = (E_6, 3)$, then $\mathrm{Stab}_G\langle v\rangle$ is not smooth if $\langle v\rangle$ has stabiliser of type F_4.*
- *(iv) If $(G, p) = (F_4, 2)$, then $\mathrm{Stab}_G\langle v\rangle$ is not smooth if $\langle v\rangle$ has stabiliser of type $B_3 \ltimes k^7$.*
- *(v) If $(G, p) = (F_4, 3)$, then G_v and $\mathrm{Stab}_G\langle v\rangle$ are not smooth if $\langle v\rangle$ has stabiliser of type $G_2T_1 \ltimes k^{14}$.*
- *(vi) If $(G, p) = (G_2, 2)$, then G_v and $\mathrm{Stab}_G\langle v\rangle$ are not smooth if $\langle v\rangle$ has stabiliser which is a long A_1-parabolic.*

Proof. Let $v \in V = V_{\min}$ and set $K := (\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$. Then certainly we have a containment $\mathrm{Lie}(K) = \mathrm{Lie}(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}} \subseteq \mathrm{Stab}_{\mathfrak{g}}\langle v\rangle =: \mathfrak{k}$. We wish to show that equality holds. For this, it suffices to show that $\dim \mathrm{Stab}_{\mathfrak{g}}\langle v\rangle = \dim(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$. The values of the right-hand side are provided by Lemma 3.1.

To prove the equality of dimensions, we work with GAP in the following way:

- (i) Construct $V_{\min}$ in GAP as in the previous section with $\mathfrak{g}$ contained in a Levi subalgebra $\mathfrak{l}$ of a parabolic $\mathfrak{p} = \mathfrak{l} + \mathfrak{q} \subseteq \mathfrak{h}$ for $\mathfrak{h} = E_6$, E_7 or E_8, such that $V_{\min}$ is contained as a quotient of the unipotent radical $\mathfrak{q}$ of $\mathfrak{p}$.
- (ii) Search for representatives for the G-orbits on V. Since one knows the finite number of orbits in case G is of type E_6, E_7 and G_2, one simply seeks this number of non-isomorphic stabilisers in G. (The papers [CC88, p467] and [LS87, Lemma 4.3] provided guidance.) For F_4, except for stabilisers of type D_4 a similar procedure works, and the stabilisers of type D_4 are described explicitly in Lemma 3.1. Representatives are given in Table 1.

Assume for the moment that G is not of type F_4 or $\langle v\rangle$ does not have stabiliser of type D_4.

G	$(\mathrm{Stab}_G\langle v\rangle)_{\mathrm{red}}$	Representative in $\mathfrak{q}$	$\dim \mathrm{Stab}_g\langle v\rangle - \dim \mathfrak{g}_v$
G_2	A_2	$e_{\alpha_1+\alpha_3+\alpha_4}$	0
	A_1-parabolic	e_{α_1}	1
F_4	B_4	$e_{\substack{112211\\1}}$	0
	$B_3T_1 \ltimes k^{14}$	e_{α_7}	1
	$G_2T_1 \ltimes k^{14}$	$e_{\alpha_7} + e_{\substack{134321\\2}}$	1
	$B_3 \ltimes k^7$	$e_{\alpha_7} + e_{\substack{122111\\1}} + e_{\substack{134321\\2}}$	0 (1 if $p = 2$)
	D_4	$e_{\substack{122111\\1}} + t \cdot e_{\substack{112211\\1}},\ t \neq 0, 1$	0
E_6	D_5-parabolic	e_{α_7}	1
	$B_4T_1 \ltimes k^{16}$	$e_{\alpha_7} + e_{\tilde{\alpha}}$	1
	F_4	$e_{\substack{122111\\1}} + e_{\substack{112211\\1}} + e_{\substack{012221\\1}}$	0 (1 if $p = 3$)
E_7	E_6-parabolic	e_{α_8}	1
	$F_4T_1 \ltimes k^{26}$	$e_{\substack{2343221\\2}} + e_{\substack{1343321\\2}} + e_{\substack{1244321\\2}}$	1
	$B_5T_1 \ltimes k^{1+32}$	$e_{\substack{2354321\\3}} + e_{\substack{2454321\\2}}$	1
	$E_6.2$	$e_{\alpha_8} + e_{\tilde{\alpha}-\alpha_8}$	0 (1 if $p = 2$)

TABLE 1. Representatives of the orbits of E_6 and E_7 on minimal modules

(iii) For each element b in a Chevalley basis B_1 of $\mathfrak{g}$, calculate the coefficients of $[b, v]$ re-expressed in terms of the Chevalley basis B of $\mathfrak{h}$.

(iv) Form the matrix M of these coefficients (which is integral, by our choice of representatives in Table 1) and calculate its elementary divisors. It turns out that unless we are in one of the exceptional cases, all elementary divisors are either 1 or 0, hence the rank r of this matrix will not change after reduction modulo p.

(v) We have $\dim \mathfrak{g} - r = \dim \mathfrak{g}_v$. If $v \in \mathfrak{g} \cdot v$ then $\dim \mathfrak{g}_v = \dim \mathrm{Stab}_{\mathfrak{g}}\langle v\rangle - 1$. Otherwise, $\dim \mathfrak{g}_v = \dim \mathrm{Stab}_{\mathfrak{g}}\langle v\rangle$. To establish which, we simply add a new line to the matrix M containing the coefficients of v in terms of B and take its elementary divisors again.

(vi) It turns out that apart from the exceptional cases $\dim \mathfrak{g}_v = \dim G_v$ and $\dim \mathrm{Stab}_{\mathfrak{g}}\langle v\rangle = \dim \mathrm{Stab}_G\langle v\rangle$.

To deal with the case where $G = F_4$ and $\langle v\rangle$ has stabiliser of type D_4, we perform a similar calculation with $\mathfrak{g}[t]$ and working with matrices over $\mathbb{Z}[t]$. Let $v = e_1 + te_2$ be a generic element in the 0-weight space of $V_{\min}$. Form M as before to get a 26×52 matrix. It turns out that 28 of these rows are identically zero and so the rank will not change after they are removed. It also turns out that for $t \neq 0, 1$, the resulting 26×24-matrix has, for any choice of t, a single non-zero entry in each row, with no two non-zero entries in a common column. Thus the rank of the matrix is 24 in all characteristics for all choices of $t \neq 0, 1$, and we are done. □

Remarks 3.3. Except when $(G, p) = (G_2, 2)$, for each exceptional case from Lemma 3.2 we have that the group-theoretic stabiliser is one fewer dimension than that in the Lie algebra. In each case, there is an extra toral element which stabilises v or $\langle v\rangle$ which is not in the Lie algebra of the reduced part of the group-theoretic stabiliser.

For $(G, p) = (G_2, 2)$, the non-smoothness of the stabiliser of the 1-space whose reduced part is an A_1-parabolic (which is a maximal smooth subgroup) implies that there is a maximal rank subalgebra $\mathfrak{h} = \mathfrak{g}_v$ containing an A_1-parabolic subalgebra of $\mathfrak{g}$ which is not the Lie algebra of any smooth subgroup of G, hence is not obtained using the Borel–de-Siebenthal algorithm. For our representative v,

$$\mathrm{Stab}_{\mathfrak{g}}\langle v\rangle = \langle e_{\alpha_2}, e_{\alpha_4}, e_{-\alpha_1}, e_{-\alpha_2}, e_{-\alpha_3}, e_{-\alpha_4}, e_{-\alpha_5}, e_{-\alpha_6}, h_1, h_2\rangle,$$

where $\langle h_1, h_2\rangle$ is a Cartan subalgebra of $\mathfrak{g}$. This is a long A_1-parabolic after throwing in the outstanding root subspace $\langle e_{\alpha_4}\rangle$ of the short $\tilde{A}_1$ subalgebra which commutes with the A_1 Levi $\langle e_{\pm\alpha_2}, h_1, h_2\rangle$.

For G_2 in characteristic 2, there are many more such maximal rank subalgebras, including one of dimension 11. These are discussed in some generality in [LH14]. One reason for the explosion in possibilities is the fact that in characteristic 2, one has, remarkably, an isomorphism of Lie algebras $\mathfrak{g} \cong \mathfrak{psl}_4$.

Finally, let us remark also that when $p = 2$, the Lie algebras $G_2T_1 \ltimes k^{14}$ and $B_3T_1 \ltimes k^7$ are isomorphic; thus, while the stabilisers of the 1-spaces in the relevant orbits in V_{26} are not isomorphic algebraic groups, their Lie algebra stabilisers are.

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e	p	Jordan Blocks
E_7	2	14^4
	3	$19^2, 18$
	5	$23^2, 10$
	7	$28, 14^2$
	19	$19^2, 18$
	23	$23^2, 10$
	others	$28, 18, 10$
$E_7(a_1)$	2	14^4
	3	$19^2, 9^2$
	7	$22, 14^2, 6$
	11	$22, 12, 11^2$
	17	$17^2, 16, 6$
	19	$19^2, 12, 6$
	others	$22, 16, 12, 6$
$E_7(a_2)$	2	$14^2, 12^2, 2^2$
	3	$18, 10^3, 8$
	13	$13^4, 4$
	17	$17^2, 10, 8, 4$
	others	$18, 16, 10, 8, 4$
$E_7(a_3)$	2	$14^2, 8^2, 6^2$
	11	$11^4, 10, 2$
	13	$13^2, 12, 10, 6, 2$
	others	$16, 12, 10^2, 6, 2$
E_6	2	$13^4, 1^4$
	3	$9^6, 1^2$
	13	$13^4, 1^4$
	others	$17^2, 9^2, 1^4$
$E_6(a_1)$	3	$9^6, 1^2$
	7	$13^2, 7^4, 1^2$
	11	$11^4, 5^2, 1^2$
	others	$13^2, 9^2, 5^2, 1^2$
D_6	2	$8^4, 6^4$
	11	$11^4, 10, 1^2$
	13	$13^2, 11^2, 6, 1^2$
	others	$16, 11^2, 10, 6, 1^2$
$E_7(a_4)$	2	$8^4, 6^4$
	3	$9^4, 7^2, 3^2$
	5	$12, 10^2, 8, 5^2, 4, 2$
	11	$11^2, 10, 8, 6, 4^2, 2$
	others	$12, 10^2, 8, 6, 4^2, 2$
$D_6(a_1)$	2	$8^4, 6^4$
	3	$9^4, 7^2, 3^2$
	5	$12, 10, 9^2, 5^2, 3^2$
	11	$11^2, 9^2, 6, 4, 3^2$
	others	$12, 10, 9^2, 6, 4, 3^2$
$D_5 + A_1$	2	$8^4, 6^2, 4^2, 2^2$
	3	$9^2, 8^4, 2^3$
	others	$11^2, 10, 8, 5^2, 2^3$
$(A_6)^{(2)}$	2	$8^4, 6^2, 4^2, 2^2$
A_6	2	7^8
	3	$9^4, 7^2, 3^2$
	7	7^8
	others	$11^2, 7^4, 3^2$
$E_7(a_5)$	2	$8^4, 6^2, 4^2, 2^2$
	3	$9^2, 8, 6^3, 4^3$
	5	$10, 8^2, 6, 5^4, 4$
	7	$7^6, 6, 4^2$
	others	$10, 8^2, 6^3, 4^3$
D_5	2	$8^4, 5^4, 1^4$
	3	$9^2, 8^4, 1^6$
	others	$11^2, 9^2, 5^2, 1^6$
$E_6(a_3)$	2	$8^4, 5^4, 1^4$
	3	$9^2, 6^4, 3^4, 1^2$
	7	$7^6, 5^2, 1^4$
	others	$9^2, 7^2, 5^4, 1^4$
$D_6(a_2)$	2	$8^2, 6^6, 2^2$
	3	$9^2, 7^2, 6, 5^2, 4^2$
	5	$10, 8, 7^2, 5^4, 4$
	7	$7^6, 5^2, 4$
	others	$10, 8, 7^2, 6, 5^2, 4^2$
$D_5(a_1) + A_1$	2	$8^2, 6^2, 4^6, 2^2$
	3	$8^3, 6^3, 3^2, 2^4$
	7	$7^6, 4, 2^5$
	others	$8^3, 6^3, 4, 2^5$
$A_5 + A_1$	2	$8^2, 6^6, 2^2$
	3	$7^2, 6^7$
	7	$7^4, 6^3, 5^2$
	others	$10, 7^2, 6^3, 5^2, 4$
$(A_5)'$	2	$7^4, 6^4, 1^4$
	3	$9^2, 6^4, 3^4, 1^2$
	7	$7^4, 6^4, 1^4$
	others	$9^2, 6^4, 5^2, 1^4$
$A_4 + A_2$	2	$7^4, 4^4, 3^4$
	3	$7^2, 6^4, 3^6$
	5	$5^{10}, 3^2$
	others	$7^4, 5^2, 3^6$
$D_5(a_1)$	2	$8^2, 5^4, 4^4, 1^4$
	7	$7^6, 3^2, 2^2, 1^4$
	others	$8^2, 7^2, 6^2, 3^2, 2^2, 1^4$
$A_4 + A_1$	5	$5^{10}, 2^2, 1^2$
	others	$7^2, 6^2, 5^2, 4^2, 3^2, 2^2, 1^2$
$D_4 + A_1$	2	$4^{12}, 2^4$
	7	$7^6, 2^5, 1^4$
	others	$8, 7^4, 6, 2^5, 1^4$
$(A_5)''$	2	$8^2, 6^6, 2^2$
	3	$7^2, 6^7$
	7	$7^2, 6^7$
	others	$10, 6^7, 4$
$A_3 + A_2 + A_1$	2	$4^{12}, 2^4$
	3	$6^3, 4^3, 3^8, 2$
	5	$5^6, 4^4, 2^5$
	others	$6^3, 4^7, 2^5$
A_4	5	$5^{10}, 1^6$
	others	$7^2, 5^6, 3^2, 1^6$
$(A_3 + A_2)^{(2)}$	2	$7^2, 5^6, 3^2, 1^6$
$A_3 + A_2$	2	$4^8, 3^8$
	3	$6^2, 5^2, 4^2, 3^8, 1^2$
	5	$5^6, 4^2, 3^4, 2^2, 1^2$
	others	$6^2, 5^2, 4^4, 3^4, 2^2, 1^2$

Table 2. Jordan Blocks of nilpotent elements on the module V_{56} for E_7, I

e	p	Jordan Blocks
$D_4(a_1)+A_1$	2	$4^{12}, 2^4$
	3	$6, 5^4, 4, 3^6, 2^4$
	5	$5^6, 4, 3^4, 2^5$
	others	$6, 5^4, 4^2, 3^4, 2^5$
D_4	2	$4^{12}, 1^8$
	others	$7^6, 1^{14}$
A_3+2A_1	2	$4^{10}, 2^8$
	5	$5^4, 4^4, 3^4, 2^3, 1^2$
	others	$6, 5^2, 4^5, 3^4, 2^3, 1^2$
$D_4(a_1)$	2	$4^{12}, 1^8$
	others	$5^6, 3^6, 1^8$
$(A_3+A_1)'$	2	$4^8, 3^4, 2^4, 1^4$
	others	$5^4, 4^4, 3^2, 2^4, 1^6$
$2A_2+A_1$	2	$4^8, 3^4, 2^4, 1^4$
	3	$3^{18}, 1^2$
	others	$5^2, 4^4, 3^6, 2^4, 1^4$
$(A_3+A_1)''$	2	$4^{10}, 2^8$
	5	$5^2, 4^8, 2^7$
	others	$6, 4^9, 2^7$
A_2+3A_1	2	$4^6, 2^{16}$
	3	$3^{14}, 2^7$
	others	$4^7, 2^{14}$
$2A_2$	2	$4^4, 3^{12}, 1^4$
	3	$3^{18}, 1^2$
	others	$5^2, 3^{14}, 1^4$
A_3	2	$4^8, 3^4, 1^{12}$
	others	$5^2, 4^8, 1^{14}$
A_2+2A_1	2	$4^4, 3^4, 2^{12}, 1^4$
	3	$3^{14}, 2^4, 1^6$
	others	$4^4, 3^6, 2^8, 1^6$
A_2+A_1	3	$3^{12}, 2^6, 1^8$
	others	$4^2, 3^8, 2^8, 1^8$
$4A_1$	2	2^{28}
	3	$3^8, 2^{13}, 1^6$
	others	$4, 3^6, 2^{14}, 1^6$
A_2	all	$3^{12}, 1^{20}$
$(3A_1)'$	2	$2^{24}, 1^8$
	others	$3^6, 2^{12}, 1^{14}$
$(3A_1)''$	2	2^{28}
	3	$3^2, 2^{25}$
	others	$4, 2^{26}$
$2A_1$	2	$2^{20}, 1^{16}$
	others	$3^2, 2^{16}, 1^{18}$
A_1	all	$2^{12}, 1^{32}$

TABLE 3. Jordan Blocks of nilpotent elements on the module V_{56} for E_7, II

e	p	Jordan Blocks
E_6	2	$13^2, 1$
	3	9^3
	13	$13^2, 1$
	others	$17, 9, 1$
$E_6(a_1)$	3	9^3
	7	$13, 7^2$
	11	$11^2, 5$
	others	$13, 9, 5$
D_5	2	$8^2, 5^2, 1$
	3	$9, 8^2, 1^2$
	others	$11, 9, 5, 1^2$
$E_6(a_3)$	2	$8^2, 5^2, 1$
	3	$9, 6^2, 3^2$
	7	$7^3, 5, 1$
	others	$9, 7, 5^2, 1$
$D_5(a_1)$	2	$8, 5^2, 4^2, 1$
	7	$7^3, 3, 2, 1$
	others	$8, 7, 6, 3, 2, 1$
A_5	2	$7^2, 6^2, 1$
	3	$9, 6^2, 3^2$
	7	$7^2, 6^2, 1$
	others	$9, 6^2, 5, 1$
$A_4 + A_1$	5	$5^5, 2$
	others	$7, 6, 5, 4, 3, 2$
D_4	2	$4^6, 1^3$
	others	$7^3, 1^6$
A_4	5	$5^5, 1^2$
	others	$7, 5^3, 3, 1^2$
$D_4(a_1)$	2	$4^6, 1^3$
	others	$5^3, 3^3, 1^3$
$A_3 + A_1$	2	$4^4, 3^2, 2^2, 1$
	others	$5^2, 4^2, 3, 2^2, 1^2$
$2A_2 + A_1$	2	$4^4, 3^2, 2^2, 1$
	3	3^9
	others	$5, 4^2, 3^3, 2^2, 1$
A_3	2	$4^4, 3^2, 1^5$
	others	$5, 4^4, 1^6$
$A_2 + 2A_1$	2	$4^2, 3^2, 2^6, 1$
	3	$3^7, 2^2, 1^2$
	others	$4^2, 3^3, 2^4, 1^2$
$2A_2$	2	$4^2, 3^6, 1$
	3	3^9
	others	$5, 3^7, 1$
$A_2 + A_1$	3	$3^6, 2^3, 1^3$
	others	$4, 3^4, 2^4, 1^3$
A_2	all	$3^6, 1^9$
$3A_1$	2	$2^{12}, 1^3$
	others	$3^3, 2^6, 1^6$
$2A_1$	2	$2^{10}, 1^7$
	others	$3, 2^8, 1^8$
A_1	all	$2^6, 1^{15}$

e	p	Jordan Blocks
F_4	2	13^2
	3	$9^2, 8$
	13	13^2
	others	$17, 9$
$F_4(a_1)$	2	$8^2, 5^2$
	3	$9, 8^2, 1$
	others	$11, 9, 5, 1$
$F_4(a_2)$	2	$8^2, 5^2$
	3	$9, 6^2, 3, 2$
	7	$7^3, 5$
	others	$9, 7, 5^2$
$(C_3)^{(2)}$	2	$7^2, 6^2$
C_3	2	$7^2, 6^2$
	3	$9, 6^2, 3, 2$
	7	$7^2, 6^2$
	others	$9, 6^2, 5$
B_3	2	$4^6, 1^2$
	others	$7^3, 1^5$
$F_4(a_3)$	2	$4^6, 1^2$
	others	$5^3, 3^3, 1^2$
$C_3(a_1)^{(2)}$	2	$4^4, 3^2, 2^2$
$C_3(a_1)$	2	$4^4, 3^2, 2^2$
	others	$5^2, 4^2, 3, 2^2, 1$
$(\tilde{A}_2 + A_1)^{(2)}$	2	$4^4, 3^2, 2^2$
$\tilde{A}_2 + A_1$	2	$4^4, 3^2, 2^2$
	3	$3^8, 2$
	others	$5, 4^2, 3^3, 2^2$
$(B_2)^{(2)}$	2	$4^4, 3^2, 1^4$
B_2	2	$4^4, 3^2, 1^4$
	others	$5, 4^4, 1^5$
$A_2 + \tilde{A}_1$	2	$4^2, 3^2, 2^6$
	3	$3^7, 2^2, 1$
	others	$4^2, 3^3, 2^4, 1$
$\tilde{A}_2$	2	$4^2, 3^6$
	3	$3^8, 2$
	others	$5, 3^7$
$(A_2)^{(2)}$	2	$3^6, 1^8$
A_2	all	$3^6, 1^8$
$A_1 + \tilde{A}_1$	2	$2^{12}, 1^2$
	others	$3^3, 2^6, 1^5$
$(\tilde{A}_1)^{(2)}$	2	$2^{10}, 1^6$
$\tilde{A}_1$	2	$2^{10}, 1^6$
	others	$3, 2^8, 1^7$
A_1	all	$2^6, 1^{14}$

e	p	Jordan Blocks
G_2	2	$4, 3$
	others	7
$G_2(a_1)$	all	$3^2, 1$
$(\tilde{A}_1)^{(3)}$	3	$3, 2^2$
$\tilde{A}_1$	2	$2^3, 1$
	others	$3, 2^2$
A_1	all	$2^2, 1^3$

TABLE 4. Jordan Blocks of nilpotent elements on the module V_{27} for E_6, V_{26} for F_4 and V_7 for G_2

Representatives of nilpotent orbits.

Orbit	Representative
G_2	$e_{10} + e_{01}$
$G_2(a_1)$	$e_{01} + e_{31}$
$(\tilde{A}_1)^{(3)}$	$e_{21} + e_{32}$
$\tilde{A}_1$	e_{10}
A_1	e_{01}

Orbit	Representative
F_4	$e_{0001} + e_{1000} + e_{0010} + e_{0100}$
$F_4(a_1)$	$e_{1000} + e_{0100} + e_{0011} + e_{0110}$
$F_4(a_2)$	$e_{0001} + e_{0011} + e_{1100} + e_{0120}$
$(C_3)^{(2)}$	$e_{0001} + e_{1110} + e_{0120} + e_{1222}$
C_3	$e_{0001} + e_{0010} + e_{0100}$
B_3	$e_{1000} + e_{0010} + e_{0100}$
$F_4(a_3)$	$e_{0100} + e_{1100} + e_{0120} + e_{1122}$
$C_3(a_1)^{(2)}$	$e_{1110} + e_{0120} + e_{0121} + e_{1222}$
$C_3(a_1)$	$e_{0100} + e_{0011} + e_{0120}$
$(\tilde{A}_2 + A_1)^{(2)}$	$e_{0111} + e_{1121} + e_{0122} + e_{1220}$
$\tilde{A}_2 + A_1$	$e_{0001} + e_{1000} + e_{0010}$
$(B_2)^{(2)}$	$e_{1100} + e_{1120} + e_{0122}$
B_2	$e_{0010} + e_{0100}$
$A_2 + \tilde{A}_1$	$e_{0001} + e_{1000} + e_{0100}$
$\tilde{A}_2$	$e_{0001} + e_{0010}$
$(A_2)^{(2)}$	$e_{1220} + e_{1122}$
A_2	$e_{1000} + e_{0100}$
$A_1 + \tilde{A}_1$	$e_{1000} + e_{0010}$
$(\tilde{A}_1)^{(2)}$	$e_{1232} + e_{2342}$
$\tilde{A}_1$	e_{0010}
A_1	e_{1000}

Table 5. Representatives of nilpotent elements for G_2 and F_4

Orbit	Representative
E_7	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$
$E_7(a_1)$	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{001000\\1}} + e_{\substack{011000\\0}}$
$E_7(a_2)$	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}} + e_{\substack{000110\\0}} + e_{\substack{000011\\0}}$
$E_7(a_3)$	$e_{\substack{100000\\0}} + e_{\substack{000001\\0}} + e_{\substack{001000\\1}} + e_{\substack{011000\\0}} + e_{\substack{000110\\0}} + e_{\substack{001100\\1}} + e_{\substack{011100\\1}}$
E_6	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}}$
$E_6(a_1)$	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{001000\\1}} + e_{\substack{011000\\0}}$
D_6	$e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$
$E_7(a_4)$	$e_{\substack{100000\\0}} + e_{\substack{000011\\0}} + e_{\substack{011000\\1}} + e_{\substack{001100\\1}} + e_{\substack{011100\\0}} + e_{\substack{001110\\0}} + e_{\substack{011110\\0}}$
$D_6(a_1)$	$e_{\substack{000000\\1}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}} + e_{\substack{001000\\1}} + e_{\substack{011000\\0}}$
$D_5 + A_1$	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$
$(A_6)^{(2)}$	$e_{\substack{000110\\0}} + e_{\substack{000011\\0}} + e_{\substack{111000\\0}} + e_{\substack{011000\\1}} + e_{\substack{001100\\1}} + e_{\substack{011100\\0}} + e_{\substack{122100\\1}}$
A_6	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$
$E_7(a_5)$	$e_{\substack{111000\\0}} + e_{\substack{011000\\1}} + e_{\substack{001100\\1}} + e_{\substack{001110\\0}} + e_{\substack{000111\\0}} + e_{\substack{011110\\0}} + e_{\substack{111110\\0}}$
D_5	$e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}}$
$E_6(a_3)$	$e_{\substack{100000\\0}} + e_{\substack{000010\\0}} + e_{\substack{110000\\0}} + e_{\substack{001000\\1}} + e_{\substack{000110\\0}} + e_{\substack{011100\\0}}$
$D_6(a_2)$	$e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{011000\\0}} + e_{\substack{001100\\0}} + e_{\substack{000110\\0}} + e_{\substack{000011\\0}}$
$D_5(a_1) + A_1$	$e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}}$

A_5+A_1	$e_{\substack{100000\\0}}+e_{\substack{000000\\1}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000010\\0}}+e_{\substack{000001\\0}}$
$(A_5)'$	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000010\\0}}$
A_4+A_2	$e_{\substack{100000\\0}}+e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000010\\0}}+e_{\substack{000001\\0}}$
$D_5(a_1)$	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{000100\\0}}+e_{\substack{001000\\1}}+e_{\substack{001100\\0}}$
A_4+A_1	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000001\\0}}$
D_4+A_1	$e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000001\\0}}$
$(A_5)''$	$e_{\substack{000000\\1}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000010\\0}}+e_{\substack{000001\\0}}$
$A_3+A_2+A_1$	$e_{\substack{100000\\0}}+e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000010\\0}}+e_{\substack{000001\\0}}$
A_4	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}$
$(A_3+A_2)^{(2)}$	$e_{\substack{000001\\0}}+e_{\substack{001000\\1}}+e_{\substack{000011\\0}}+e_{\substack{011100\\0}}+e_{\substack{001110\\0}}+e_{\substack{011110\\1}}$
A_3+A_2	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000010\\0}}+e_{\substack{000001\\0}}$
$D_4(a_1)+A_1$	$e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{000001\\0}}+e_{\substack{001000\\1}}+e_{\substack{001100\\0}}$
D_4	$e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}$
A_3+2A_1	$e_{\substack{100000\\0}}+e_{\substack{000000\\1}}+e_{\substack{001000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000001\\0}}$
$D_4(a_1)$	$e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{001000\\1}}+e_{\substack{001100\\0}}$
$(A_3+A_1)'$	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000010\\0}}$
$2A_2+A_1$	$e_{\substack{100000\\0}}+e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000010\\0}}$
$(A_3+A_1)''$	$e_{\substack{000000\\1}}+e_{\substack{000100\\0}}+e_{\substack{000010\\0}}+e_{\substack{000001\\0}}$
A_2+3A_1	$e_{\substack{100000\\0}}+e_{\substack{000000\\1}}+e_{\substack{010000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000001\\0}}$
$2A_2$	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{000010\\0}}+e_{\substack{000001\\0}}$
A_3	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{001000\\0}}$
A_2+2A_1	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{000100\\0}}+e_{\substack{000001\\0}}$
A_2+A_1	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}+e_{\substack{000100\\0}}$
$4A_1$	$e_{\substack{100000\\0}}+e_{\substack{000000\\1}}+e_{\substack{000100\\0}}+e_{\substack{000001\\0}}$
A_2	$e_{\substack{100000\\0}}+e_{\substack{010000\\0}}$
$(3A_1)'$	$e_{\substack{100000\\0}}+e_{\substack{001000\\0}}+e_{\substack{000010\\0}}$
$(3A_1)''$	$e_{\substack{000000\\1}}+e_{\substack{000100\\0}}+e_{\substack{000001\\0}}$
$2A_1$	$e_{\substack{100000\\0}}+e_{\substack{001000\\0}}$
A_1	$e_{\substack{100000\\0}}$

Table 7: Representatives of nilpotent elements for E_7

Orbit	Representative
E_8	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}$
$E_8(a_1)$	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0110000\\0}}$
$E_8(a_2)$	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0000001\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0011000\\0}}+e_{\substack{0001100\\0}}+e_{\substack{0000110\\0}}$
$E_8(a_3)$	$e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}+e_{\substack{1100000\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0110000\\0}}+e_{\substack{0011000\\0}}+e_{\substack{0001100\\0}}+e_{\substack{0111000\\0}}$
$E_8(a_4)$	$e_{\substack{1100000\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0110000\\0}}+e_{\substack{0011000\\0}}+e_{\substack{0001100\\0}}+e_{\substack{0000110\\0}}+e_{\substack{0000011\\0}}+e_{\substack{0111000\\0}}$
E_7	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}$
$E_8(b_4)$	$e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}+e_{\substack{1100000\\0}}+e_{\substack{0110000\\1}}+e_{\substack{0011000\\1}}+e_{\substack{0111000\\0}}+e_{\substack{0011100\\0}}+e_{\substack{0001110\\0}}$

$E_8(a_5)$	$e_{\substack{1100000\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0000011\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}} + e_{\substack{0011100\\0}} + e_{\substack{0011100\\1}}$
$E_7(a_1)$	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$
$E_8(b_5)$	$e_{\substack{0000001\\0}} + e_{\substack{1110000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{1111000\\0}} + e_{\substack{0111100\\1}}$
$(D_7)^{(2)}$	$e_{\substack{1000000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{0000111\\0}} + e_{\substack{1111111\\1}}$
D_7	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$
$E_8(a_6)$	$e_{\substack{0000110\\0}} + e_{\substack{0000011\\0}} + e_{\substack{1110000\\0}} + e_{\substack{0011100\\0}} + e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} + e_{\substack{0111000\\1}} + e_{\substack{1111100\\1}}$
$E_7(a_2)$	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}}$
$E_6 + A_1$	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$
$(D_7(a_1))^{(2)}$	$e_{\substack{0001000\\0}} + e_{\substack{0000001\\0}} + e_{\substack{1100000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0000011\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0111100\\0}} + e_{\substack{0121110\\1}}$
$D_7(a_1)$	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$
$E_8(b_6), p > 2$	$e_{\substack{0000011\\0}} + e_{\substack{0001111\\0}} + e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} + e_{\substack{0111000\\1}} + e_{\substack{0011100\\1}} + e_{\substack{0111100\\0}} + e_{\substack{0011110\\0}}$
$E_8(b_6), p = 2$	$e_{\substack{0000011\\0}} + e_{\substack{0000111\\0}} + e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} + e_{\substack{0111000\\1}} + e_{\substack{0011100\\1}} + e_{\substack{0111100\\0}} + e_{\substack{0011110\\0}}$
$E_7(a_3)$	$e_{\substack{1000000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\1}}$
$E_6(a_1) + A_1$	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$
$A_7^{(3)}$	$e_{\substack{0001110\\0}} + e_{\substack{0000111\\0}} + e_{\substack{1110000\\1}} + e_{\substack{1111000\\0}} + e_{\substack{0011100\\1}} + e_{\substack{0111100\\0}} + e_{\substack{0121000\\1}} + e_{\substack{0011111\\0}}$
A_7	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$
$D_7(a_2)$	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}}$
E_6	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}}$
D_6	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$
$(D_5 + A_2)^{(2)}$	$e_{\substack{0000011\\0}} + e_{\substack{0000111\\0}} + e_{\substack{1111000\\1}} + e_{\substack{1111100\\0}} + e_{\substack{0121000\\1}} + e_{\substack{0111100\\1}} + e_{\substack{0011110\\1}} + e_{\substack{0111110\\0}}$
$D_5 + A_2$	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$
$E_6(a_1)$	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$
$E_7(a_4)$	$e_{\substack{1000000\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}} + e_{\substack{0011100\\0}} + e_{\substack{0111100\\0}}$
$A_6 + A_1$	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$
$D_6(a_1)$	$e_{\substack{0000000\\1}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$
$(A_6)^{(2)}$	$e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}} + e_{\substack{1110000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}} + e_{\substack{1221000\\1}}$
A_6	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$
$E_8(a_7)$	$e_{\substack{1111000\\1}} + e_{\substack{0111100\\1}} + e_{\substack{0111110\\0}} + e_{\substack{0011111\\0}} + e_{\substack{0121100\\1}} + e_{\substack{1121100\\1}} + e_{\substack{1121110\\1}} + e_{\substack{1121111\\1}}$
$D_5 + A_1$	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}}$
$E_7(a_5)$	$e_{\substack{1110000\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0011100\\0}} + e_{\substack{0001110\\0}} + e_{\substack{0111100\\0}} + e_{\substack{1111100\\0}}$
$E_6(a_3) + A_1$	$e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}} + e_{\substack{1100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0001100\\0}} + e_{\substack{0111000\\0}}$
$D_6(a_2)$	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}}$
$D_5(a_1) + A_2$	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}}$
$A_5 + A_1$	$e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$
$A_4 + A_3$	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$
D_5	$e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}}$
$E_6(a_3)$	$e_{\substack{1000000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{1100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0001100\\0}} + e_{\substack{0111000\\0}}$
$(D_4 + A_2)^{(2)}$	$e_{\substack{0010000\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0000110\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0001110\\0}} + e_{\substack{0111100\\0}}$
$D_4 + A_2$	$e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$

$A_4+A_2+A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}$
$D_5(a_1)+A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0011000\\0}}$
A_5	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}$
A_4+A_2	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}$
A_4+2A_1	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}$
$D_5(a_1)$	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0011000\\0}}$
$2A_3$	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}$
A_4+A_1	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}$
$D_4(a_1)+A_2$	$e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0000001\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0011000\\0}}$
D_4+A_1	$e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}$
$A_3+A_2+A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}$
A_4	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0001000\\0}}$
$(A_3+A_2)^{(2)}$	$e_{\substack{0000010\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0000110\\0}}+e_{\substack{0111000\\0}}+e_{\substack{0011100\\0}}+e_{\substack{0111100\\1}}$
A_3+A_2	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000010\\0}}$
$D_4(a_1)+A_1$	$e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0011000\\0}}$
A_3+2A_1	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000001\\0}}$
$2A_2+2A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000001\\0}}$
D_4	$e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0001000\\0}}$
$D_4(a_1)$	$e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0010000\\1}}+e_{\substack{0011000\\0}}$
A_3+A_1	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0000100\\0}}$
$2A_2+A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000001\\0}}$
$2A_2$	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000100\\0}}$
A_2+3A_1	$e_{\substack{1000000\\0}}+e_{\substack{0000000\\1}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}$
A_3	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0010000\\0}}$
A_2+2A_1	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}+e_{\substack{0000010\\0}}$
A_2+A_1	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}+e_{\substack{0001000\\0}}$
$4A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0000100\\0}}+e_{\substack{0000001\\0}}$
A_2	$e_{\substack{1000000\\0}}+e_{\substack{0100000\\0}}$
$3A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0010000\\0}}+e_{\substack{0000100\\0}}$
$2A_1$	$e_{\substack{1000000\\0}}+e_{\substack{0010000\\0}}$
A_1	$e_{\substack{1000000\\0}}$

Table 8: Representatives of nilpotent elements for E_8

Orbit	Representative
E_6	$e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$
$E_6(a_1)$	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}} + e_{\substack{00100\\1}} + e_{\substack{01100\\0}}$
D_5	$e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}}$
$E_6(a_3)$	$e_{\substack{10000\\0}} + e_{\substack{00100\\1}} + e_{\substack{01100\\0}} + e_{\substack{00011\\0}} + e_{\substack{00110\\1}} + e_{\substack{01110\\1}}$
$D_5(a_1)$	$e_{\substack{00000\\1}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}} + e_{\substack{00100\\1}} + e_{\substack{01100\\0}}$
A_5	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$
$A_4 + A_1$	$e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$
D_4	$e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}}$
A_4	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}}$
$D_4(a_1)$	$e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}}$
$A_3 + A_1$	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$
$2A_2 + A_1$	$e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$
A_3	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}}$
$A_2 + 2A_1$	$e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00001\\0}}$
$2A_2$	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$
$A_2 + A_1$	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}}$
A_2	$e_{\substack{10000\\0}} + e_{\substack{01000\\0}}$
$3A_1$	$e_{\substack{10000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$
$2A_1$	$e_{\substack{10000\\0}} + e_{\substack{00100\\0}}$
A_1	$e_{\substack{10000\\0}}$

TABLE 6. Representatives of nilpotent elements for E_6

3.1. **Adjoint Jordan blocks.**

e	p	Jordan Blocks
E_8	2	$16^8, 15^8$
	3	$27^7, 26^2, 3, 2^2$
	5	$25^8, 24^2$
	7	$49^2, 47, 35, 27, 23, 15, 3$
	19	$59, 47, 39, 35, 27, 19^2, 3$
	29	$59, 47, 39, 29^2, 27, 15, 3$
	31	31^8
	37	$37^6, 23, 3$
	41	$41^4, 39, 27, 15, 3$
	43	$43^4, 35, 23, 15, 3$
	47	$47^3, 39, 27, 23, 15, 3$
	53	$53^2, 39, 35, 27, 23, 15, 3$
	others	$59, 47, 39, 35, 27, 23, 15, 3$
$E_8(a_1)$	2	$16^8, 15^8$
	3	$27^7, 26^2, 3, 2^2$
	5	$25^8, 24^2$
	7	$47, 39, 35, 28^2, 21^2, 15, 11, 3$
	13	$47, 39, 35, 27, 26^2, 19, 13^2, 3$
	17	$47, 35, 34^2, 27, 17^4, 3$
	19	$39, 38^2, 19^7$
	23	$47, 39, 35, 29, 23^3, 15, 11, 3$
	29	$29^7, 27, 15, 3$
	31	$31^6, 29, 19, 11, 3$
	37	$37^4, 29, 23, 19, 15, 11, 3$
	41	$41^2, 39, 29, 27, 23, 19, 15, 11, 3$
	43	$43^2, 35, 29, 27, 23, 19, 15, 11, 3$
	others	$47, 39, 35, 29, 27, 23, 19, 15, 11, 3$
$E_8(a_2)$	2	$16^8, 15^8$
	3	$27^5, 26^2, 20^2, 9^2, 3$
	5	$25^6, 24^2, 20^2, 5^2$
	7	$39, 35, 28^2, 23, 21^2, 15, 14^2, 7, 3$
	13	$39, 27, 26^4, 13^6$
	17	$35, 34^2, 23, 17^7, 3$
	19	$39, 35, 29, 27, 23, 19^3, 17, 11, 7, 3$
	23	$23^{10}, 15, 3$
	29	$29^5, 27, 23, 17, 15, 11, 7, 3$
	31	$31^4, 29, 23, 19, 17, 15, 11, 7, 3$
	37	$37^2, 29, 27, 23^2, 19, 17, 15, 11, 7, 3$
	others	$39, 35, 29, 27, 23^2, 19, 17, 15, 11, 7, 3$
$E_8(a_3)$	2	$16^8, 15^8$
	3	$27^3, 20^2, 18^4, 9^4, 8^2, 3$
	5	$25^4, 24^2, 22^2, 15^2, 10^2, 3^2$
	7	$35, 28^2, 21^4, 14^4, 7^2, 3$
	13	$35, 29, 27^2, 23, 19^2, 17, 13^2, 11, 9, 3^2$
	17	$35, 29, 27^2, 23, 19, 17^3, 11^2, 9, 3^2$
	19	$19^{12}, 17, 3$
	23	$23^9, 15, 11, 9, 3^2$
	29	$29^3, 27^2, 19^2, 17, 15, 11^2, 9, 3^2$
	31	$31^2, 29, 27, 23, 19^2, 17, 15, 11^2, 9, 3^2$
	others	$35, 29, 27^2, 23, 19^2, 17, 15, 11^2, 9, 3^2$
$E_8(a_4)$	2	$16^8, 15^8$
	3	$27, 26^2, 21^2, 18^2, 15^2, 12^2, 9^2, 8^2, 3$
	5	$25^2, 24^2, 20^2, 15^4, 10^4, 5^2$
	7	$29, 27, 21^4, 15^3, 14^2, 7^5$
	13	$27, 26^2, 23, 19, 15, 13^8, 5, 3$
	17	$17^{13}, 15, 9, 3$
	19	$19^{10}, 17, 15, 11, 7, 5, 3$
	23	$23^6, 19, 15^3, 11^2, 9, 7, 5, 3$
	others	$29, 27, 23^2, 19^2, 17, 15^3, 11^2, 9, 7, 5, 3$

e	p	Jordan Blocks
E_7	2	$16^6, 15^2, 14^8, 2^2, 1^6$
	3	$27^3, 19^4, 18^2, 9^4, 8^2, 1^3$
	5	$25^4, 23^4, 15^2, 10^2, 3, 1^3$
	7	$35, 28^2, 21^4, 14^4, 7^2, 1^3$
	13	$35, 28^2, 27, 23, 19, 18^2, 13^2, 10^2, 3, 1^3$
	17	$35, 28^2, 27, 23, 18^2, 17^2, 11, 10^2, 3, 1^3$
	19	$19^{11}, 18^2, 1^3$
	23	$23^9, 15, 10^2, 3, 1^3$
	29	$29^2, 28^2, 27, 19, 18^2, 15, 11, 10^2, 3, 1^3$
	31	$31^2, 28^2, 23, 19, 18^2, 15, 11, 10^2, 3, 1^3$
	others	$35, 28^2, 27, 23, 19, 18^2, 15, 11, 10^2, 3, 1^3$
$E_8(b_4)$	2	$16^6, 15^2, 14^8, 2^2, 1^6$
	3	$27, 20^2, 19, 18^2, 17^3, 9^8, 3$
	5	$25^2, 23, 21, 19, 17^2, 15^2, 13, 11^3, 7^2, 5, 3^2$
	7	$27, 23, 21^3, 15, 14^6, 7^4, 5, 3$
	11	$23, 22^4, 13, 11^{11}, 3$
	13	$27, 23^2, 21, 19, 17^2, 15, 13^3, 11^2, 7^2, 5, 3^2$
	17	$17^{12}, 15, 11, 7, 5, 3^2$
	19	$19^9, 17, 13, 11^2, 7^2, 5, 3^2$
	23	$23^4, 21, 17^2, 15^2, 13, 11^3, 7^2, 5, 3^2$
	others	$27, 23^2, 21, 19, 17^2, 15^2, 13, 11^3, 7^2, 5, 3^2$
$E_8(a_5)$	2	$16^4, 15^4, 12^8, 4^4, 3^4$
	3	$23^2, 21, 19, 17, 15^3, 12^4, 11^2, 9, 7, 5, 3^3$
	5	$23^2, 21, 19, 17, 15^3, 13^2, 11^3, 10^2, 7, 5, 3^3$
	7	$23, 21^3, 15, 14^6, 11, 7^7, 3$
	11	$23, 22^2, 19, 17, 15^2, 13, 11^8, 5, 3^3$
	13	$13^{18}, 11, 3$
	17	$17^9, 15^2, 13, 11^2, 9, 7, 5, 3^3$
	19	$19^7, 15, 13^2, 11^4, 9, 7, 5, 3^3$
	others	$23^2, 21, 19, 17, 15^3, 13^2, 11^4, 9, 7, 5, 3^3$
$E_7(a_1)$	2	$16^6, 15^2, 14^8, 2^2, 1^6$
	3	$27, 19^5, 17^3, 9^8, 1^3$
	5	$25^2, 22^2, 19, 17, 16^2, 15, 12^2, 11^2, 7, 6^2, 3, 1^3$
	7	$27, 22^2, 21^2, 15, 14^6, 7^3, 6^2, 1^3$
	11	$23, 22^4, 12^2, 11^{10}, 1^3$
	13	$27, 23, 22^2, 19, 17, 16^2, 13^2, 12^2, 11, 7, 6^2, 3, 1^3$
	17	$17^{11}, 16^2, 11, 6^2, 3, 1^3$
	19	$19^9, 17, 12^2, 11, 7, 6^2, 3, 1^3$
	23	$23^3, 22^2, 17, 16^2, 15, 12^2, 11^2, 7, 6^2, 3, 1^3$
	others	$27, 23, 22^2, 19, 17, 16^2, 15, 12^2, 11^2, 7, 6^2, 3, 1^3$
$E_8(b_5)$	2	$16^4, 15^4, 12^8, 4^4, 3^4$
	3	$19, 18^2, 17^3, 11^3, 9^{11}, 7, 3$
	5	$23, 19^2, 17^3, 15^3, 11, 10^4, 9, 7^2, 5, 3^4$
	13	$13^{18}, 5, 3^3$
	17	$17^9, 15, 11^2, 9^3, 7^2, 5, 3^4$
	19	$19^4, 17^3, 15^2, 11^3, 9^3, 7^2, 5, 3^4$
	others	$23, 19^2, 17^3, 15^3, 11^3, 9^3, 7^2, 5, 3^4$
$(D_7)^{(2)}$	2	$16^2, 15^6, 8^{14}, 7^2$
D_7	2	$8^{24}, 7^8$
	7	$22^2, 21^2, 15, 14^4, 13^3, 7^7, 1^3$
	11	$23, 22^2, 19, 16^2, 13^3, 11^7, 4^2, 3, 1^3$
	13	$13^{17}, 12^2, 1^3$
	17	$17^8, 16^2, 13^3, 10^2, 7, 4^2, 3, 1^3$
	19	$19^7, 13^3, 12^2, 11, 10^2, 7, 4^2, 3, 1^3$
	others	$23, 22^2, 19, 16^2, 15, 13^3, 12^2, 11, 10^2, 7, 4^2, 3, 1^3$

TABLE 9. Jordan Blocks of nilpotent elements on the adjoint module for E_8, I

e	p	Jordan Blocks
$E_8(a_6)$	2	$16^2, 15^6, 8^{14}, 7^2$
	3	$19, 18^2, 15^3, 13^3, 9^9, 7, 6^2, 3^3$
	5	$19^2, 15^5, 11, 10^8, 9, 5^7$
	7	$19^2, 17, 15, 14^4, 13, 11^3, 9^2, 7^7, 3^3$
	11	$11^{21}, 7^2, 3$
	13	$13^{15}, 9^2, 7^3, 5, 3^3$
	17	$17^5, 15, 13^3, 11^3, 9^3, 7^5, 5, 3^3$
	others	$19^2, 17, 15^3, 13^3, 11^3, 9^3, 7^5, 5, 3^3$
$E_7(a_2)$	2	$16^4, 14^4, 13^4, 12^4, 4^2, 3^2, 2^6, 1^2$
	3	$19, 18^2, 17^3, 10^6, 9^7, 8^2, 1^3$
	5	$23, 19, 18^2, 17, 16^2, 15^2, 11, 10^4, 8^2, 7, 4^2, 3^2, 1^3$
	13	$13^{18}, 4^2, 3, 1^3$
	17	$17^9, 15, 11, 10^2, 9, 8^2, 7, 4^2, 3^2, 1^3$
	19	$19^3, 18^2, 17, 16^2, 15, 11^2, 10^2, 9, 8^2, 7, 4^2, 3^2, 1^3$
	others	$23, 19, 18^2, 17, 16^2, 15^2, 11^2, 10^2, 9, 8^2, 7, 4^2, 3^2, 1^3$
$E_6 + A_1$	2	$16^4, 14^4, 13^4, 12^4, 4^2, 3^2, 2^6, 1^2$
	3	$9^{25}, 8^2, 3, 2^2$
	13	$13^{18}, 3, 2^4, 1^3$
	17	$17^9, 15, 10^2, 9^3, 8^2, 3^2, 2^4, 1^3$
	19	$19^2, 18^2, 17^3, 16^2, 11, 10^2, 9^3, 8^2, 3^2, 2^4, 1^3$
	others	$23, 18^2, 17^3, 16^2, 15, 11, 10^2, 9^3, 8^2, 3^2, 2^4, 1^3$
$(D_7(a_1))^{(2)}$	2	$16^2, 15^2, 14^4, 9^4, 8^8, 6^4, 2^2, 1^2$
$D_7(a_1)$	2	$8^{24}, 7^8$
	3	$19, 17^2, 15^3, 12^2, 11^4, 9^5, 6^4, 3^4, 1$
	5	$19, 17^2, 15^3, 13, 11^3, 10^6, 7^2, 5^4, 3^3, 1$
	11	$11^{20}, 9^2, 3^3, 1$
	13	$13^{13}, 11^3, 9, 7^2, 5^2, 3^4, 1$
	17	$17^4, 15^2, 13, 11^6, 9^3, 7^3, 5^2, 3^4, 1$
	others	$19, 17^2, 15^3, 13, 11^6, 9^3, 7^3, 5^2, 3^4, 1$
$E_8(b_6)$	2	$16^2, 14^2, 13^2, 12^2, 11^2, 10^2, 9^2, 8^4, 6^2, 5^2, 4^4, 2^4$
	3	$9^{25}, 8^2, 1^7$
	5	$17, 15^3, 13^2, 11^3, 10^6, 7^3, 5^8, 3^2$
	7	$15, 14^6, 11^2, 7^{17}, 5, 3$
	11	$11^{18}, 9, 7^2, 5^3, 3^4$
	13	$13^{10}, 11^3, 9^2, 7^5, 5^4, 3^4$
	others	$17, 15^3, 13^2, 11^6, 9^3, 7^5, 5^4, 3^4$
$E_7(a_3)$	2	$16^2, 15^2, 14^4, 9^4, 8^8, 6^4, 2^2, 1^2$
	3	$19, 17, 16^2, 15^2, 12^2, 11^2, 10^4, 9^3, 6^4, 3^2, 2^2, 1^3$
	5	$19, 17, 16^2, 15^2, 12^2, 11^2, 10^6, 7, 6^2, 5^3, 3, 2^2, 1^3$
	11	$11^{19}, 10^2, 9, 3, 2^2, 1^3$
	13	$13^{12}, 12^2, 11, 10^2, 7, 6^2, 5, 3^2, 2^2, 1^3$
	17	$17^3, 16^2, 15, 12^2, 11^3, 10^4, 9, 7^2, 6^2, 5, 3^2, 2^2, 1^3$
	others	$19, 17, 16^2, 15^2, 12^2, 11^3, 10^4, 9, 7^2, 6^2, 5, 3^2, 2^2, 1^3$
$E_6(a_1) + A_1$	2	$16^2, 14^2, 13^2, 12^2, 11^2, 10^2, 9^2, 8^4, 6^2, 5^2, 4^4, 2^4$
	3	$9^{25}, 8^2, 3, 2^2$
	5	$17, 15, 14^2, 13^2, 12^2, 11, 10^4, 9^2, 8^2, 7, 5^7, 3^2, 2^2, 1$
	7	$15, 14^4, 13^2, 12^2, 7^{17}, 3, 2^2, 1$
	11	$11^{18}, 9, 6^2, 5^2, 4^2, 3^2, 2^2, 1$
	13	$13^{10}, 11, 10^2, 9^2, 8^2, 7, 6^2, 5^3, 4^2, 3^2, 2^2, 1$
	others	$17, 15, 14^2, 13^2, 12^2, 11^2, 10^2, 9^3, 8^2, 7, 6^2, 5^3, 4^2, 3^2, 2^2, 1$
$A_7^{(3)}$	3	$9^{25}, 8^2, 3^1, 2^2$
A_7	2	$8^{24}, 7^8$
	3	$9^{25}, 8^2, 1^7$
	5	$16^2, 15, 13^3, 12^2, 11, 10^2, 9^3, 8^4, 7, 5^7, 3, 1^3$
	7	$15, 14^4, 13^3, 8^4, 7^{13}, 6^2, 1^3$
	11	$11^{17}, 8^4, 5^3, 4^2, 3, 1^3$
	13	$13^9, 12^2, 9^3, 8^4, 7, 6^2, 5^3, 4^2, 3, 1^3$
	others	$16^2, 15, 13^3, 12^2, 11, 10^2, 9^3, 8^4, 7, 6^2, 5^3, 4^2, 3, 1^3$
$D_7(a_2)$	2	$8^{24}, 7^8$
	3	$9^{23}, 7, 5^6, 3, 1$
	5	$15, 14^2, 11, 10^{10}, 9^2, 5^{15}, 1$
	7	$15, 14^2, 13, 12^2, 11^2, 10^2, 9^2, 8^2, 7^9, 5^2, 4^2, 3^2, 2^2, 1$
	11	$11^{14}, 9^2, 8^2, 7^2, 6^2, 5^3, 4^2, 3^2, 2^2, 1$
	13	$13^7, 11, 10^2, 9^3, 8^4, 7^3, 6^2, 5^3, 4^2, 3^2, 2^2, 1$
	others	$15, 14^2, 13, 12^2, 11^2, 10^2, 9^3, 8^4, 7^3, 6^2, 5^3, 4^2, 3^2, 2^2, 1$

TABLE 10. Jordan Blocks of nilpotent elements on the adjoint module for E_8, II

e	p	Jordan Blocks
E_6	2	$16^4, 13^{12}, 4^2, 3^2, 1^{14}$
	3	$9^{25}, 8^2, 1^7$
	13	$13^{18}, 1^{14}$
	17	$17^9, 15, 9^7, 3, 1^{14}$
	19	$19^2, 17^7, 11, 9^7, 3, 1^{14}$
	others	$23, 17^7, 15, 11, 9^7, 3, 1^{14}$
D_6	2	$8^{22}, 7^2, 6^8, 2^2, 1^6$
	3	$19, 16^4, 15, 11^5, 10^4, 9^2, 6^4, 3, 1^{10}$
	5	$19, 16^4, 15, 11^6, 10^4, 6^4, 5^2, 1^{10}$
	11	$11^{18}, 10^4, 1^{10}$
	13	$13^{12}, 11^5, 6^4, 3, 1^{10}$
	17	$17^2, 16^4, 11^6, 10^4, 7, 6^4, 3, 1^{10}$
	others	$19, 16^4, 15, 11^6, 10^4, 7, 6^4, 3, 1^{10}$
$(D_5+A_2)^{(2)}$	2	$8^{22}, 7^2, 6^8, 2^2, 1^6$
D_5+A_2	2	$8^{20}, 7^4, 4^{12}, 3^4$
	3	$9^{19}, 8^4, 6^4, 3^7$
	5	$15, 13^2, 11^3, 10^8, 9, 7^2, 5^{11}, 3^5, 1$
	11	$11^{13}, 9^3, 7^4, 5^5, 3^8, 1$
	13	$13^4, 11^6, 9^5, 7^5, 5^5, 3^8, 1$
	others	$15, 13^2, 11^7, 9^5, 7^5, 5^5, 3^8, 1$
$E_6(a_1)$	2	$16^2, 13^6, 11^2, 9^6, 8^2, 5^6, 4^2, 1^8$
	3	$9^{25}, 8^2, 1^7$
	5	$17, 15, 13^6, 11, 10^2, 9^6, 7, 5^7, 3, 1^8$
	7	$15, 14^2, 13^6, 7^{17}, 1^8$
	11	$11^{18}, 9, 5^6, 3, 1^8$
	13	$13^{10}, 11, 9^6, 7, 5^7, 3, 1^8$
	others	$17, 15, 13^6, 11^2, 9^7, 7, 5^7, 3, 1^8$
$E_7(a_4)$	2	$8^{22}, 7^2, 6^8, 2^2, 1^6$
	3	$9^{19}, 8^2, 7^4, 6^2, 3^6, 1^3$
	5	$15, 13, 12^2, 11^2, 10^8, 8^2, 7, 5^{10}, 4^2, 3^2, 2^2, 1^3$
	11	$11^{12}, 10^2, 9, 8^2, 7^2, 6^2, 5^2, 4^4, 3^4, 2^2, 1^3$
	13	$13^3, 12^2, 11^3, 10^4, 9^2, 8^2, 7^3, 6^2, 5^2, 4^4, 3^4, 2^2, 1^3$
	others	$15, 13, 12^2, 11^4, 10^4, 9^2, 8^2, 7^3, 6^2, 5^2, 4^4, 3^4, 2^2, 1^3$
A_6+A_1	2	$8^{22}, 7^2, 6^8, 2^2, 1^6$
	3	$9^{19}, 8^2, 7^4, 6^2, 3^6, 1^3$
	7	$7^{35}, 3$
	11	$11^{11}, 8^4, 7^5, 6^4, 5^3, 4^2, 3^2, 2^2, 1^3$
	others	$13^3, 12^2, 11, 10^2, 9^3, 8^4, 7^5, 6^4, 5^3, 4^2, 3^2, 2^2, 1^3$
$D_6(a_1)$	2	$8^{22}, 7^2, 6^8, 2^2, 1^6$
	3	$9^{19}, 7^8, 3^5, 1^6$
	5	$15, 12^4, 11, 10^6, 9^4, 7, 5^{10}, 3^5, 1^6$
	11	$11^{12}, 9^5, 7, 6^4, 4^4, 3^6, 1^6$
	13	$13^2, 12^4, 11, 10^4, 9^5, 7^2, 6^4, 4^4, 3^6, 1^6$
	others	$15, 12^4, 11^2, 10^4, 9^5, 7^2, 6^4, 4^4, 3^6, 1^6$
$(A_6)^{(2)}$	2	$8^{22}, 7^2, 6^8, 2^2, 1^6$
A_6	2	$8^{14}, 7^{18}, 1^{10}$
	3	$9^{19}, 7^8, 3^5, 1^6$
	7	$7^{35}, 1^3$
	11	$11^{11}, 7^{13}, 5^3, 3^5, 1^6$
	others	$13^3, 11^5, 9^3, 7^{13}, 5^3, 3^5, 1^6$
$E_8(a_7)$	2	$8^{20}, 7^4, 4^{12}, 3^4$
	3	$9^{14}, 7^2, 6^8, 5^6, 3^{10}$
	5	$11, 10^6, 9^3, 7^5, 5^{20}, 3^5$
	7	$7^{30}, 5^4, 3^6$
	others	$11^4, 9^6, 7^{10}, 5^{10}, 3^{10}$
D_5+A_1	2	$8^{20}, 6^4, 5^4, 4^6, 3^2, 2^6, 1^2$
	3	$9^{15}, 8^8, 7^4, 3, 2^6, 1^6$
	5	$15, 12^2, 11^5, 10^4, 9^3, 8^2, 5^{10}, 3, 2^6, 1^6$
	11	$11^{11}, 10^2, 9^3, 8^2, 6^2, 5^4, 4^2, 3^2, 2^6, 1^6$
	13	$13^2, 12^2, 11^4, 10^4, 9^3, 8^2, 7, 6^2, 5^4, 4^2, 3^2, 2^6, 1^6$
	others	$15, 12^2, 11^5, 10^4, 9^3, 8^2, 7, 6^2, 5^4, 4^2, 3^2, 2^6, 1^6$
$E_7(a_5)$	2	$8^{20}, 6^4, 5^4, 4^6, 3^2, 2^6, 1^2$
	3	$9^{13}, 8^2, 7, 6^8, 5^3, 4^6, 3^6, 1^3$
	5	$11, 10^6, 9, 8^4, 7^2, 6^2, 5^{18}, 4^2, 3^3, 1^3$
	7	$7^{29}, 6^2, 5, 4^4, 3^3, 1^3$
	others	$11^3, 10^2, 9^3, 8^4, 7^5, 6^6, 5^4, 4^6, 3^6, 1^3$
$E_6(a_3)+A_1$	2	$8^{20}, 6^4, 5^4, 4^6, 3^2, 2^6, 1^2$
	3	$9^9, 8^2, 6^{16}, 3^{17}, 2^2$
	5	$11, 10^4, 9^3, 8^4, 7^3, 6^2, 5^{17}, 3^3, 2^4, 1^3$
	7	$7^{28}, 6^2, 5^3, 4^2, 3^2, 2^4, 1^3$
	others	$11^2, 10^2, 9^4, 8^4, 7^4, 6^6, 5^7, 4^4, 3^4, 2^4, 1^3$

TABLE 11. Jordan Blocks of nilpotent elements on the adjoint module for E_8, III

$D_6(a_2)$	2	$8^{12}, 7^4, 6^{16}, 2^{12}, 1^4$
	3	$9^{13}, 7^5, 6^4, 5^5, 4^8, 3^3, 1^6$
	5	$11, 10^6, 8^4, 7^5, 5^{17}, 4^4, 3, 1^6$
	7	$7^{29}, 5^4, 4^4, 3, 1^6$
	others	$11^2, 10^4, 9, 8^4, 7^7, 6^4, 5^5, 4^8, 3^3, 1^6$
$D_5(a_1) + A_2$	2	$8^{10}, 7^6, 4^{30}, 3^2$
	3	$9^9, 8^2, 6^{16}, 3^{17}, 2^2$
	5	$11, 10^2, 9^3, 8^6, 7^4, 6^4, 5^{10}, 4^2, 3^7, 2^4, 1^3$
	7	$7^{28}, 5^3, 4^2, 3^6, 2^4, 1^3$
	others	$11, 10^2, 9^3, 8^6, 7^4, 6^6, 5^6, 4^4, 3^7, 2^4, 1^3$
$A_5 + A_1$	2	$8^{12}, 7^4, 6^{16}, 2^{12}, 1^4$
	3	$9^9, 7^4, 6^{14}, 3^{16}, 2^2, 1^3$
	5	$11, 10^4, 9^3, 8^2, 7^4, 6^6, 5^{13}, 4^2, 3, 2^4, 1^6$
	7	$7^{25}, 6^6, 5^4, 3, 2^4, 1^6$
	others	$11, 10^4, 9^3, 8^2, 7^5, 6^8, 5^7, 4^4, 3^2, 2^4, 1^6$
$A_4 + A_3$	2	$8^{10}, 7^6, 4^{30}, 3^2$
	3	$9^7, 8^2, 7^6, 6^6, 5^6, 4^6, 3^{10}, 2^2, 1^3$
	5	$5^{48}, 4^2$
	7	$7^{24}, 6^2, 5^3, 4^6, 3^6, 2^4, 1^3$
	others	$10^2, 9^3, 8^4, 7^6, 6^6, 5^6, 4^8, 3^6, 2^4, 1^3$
D_5	2	$8^{20}, 5^{12}, 4^2, 3^2, 1^{14}$
	3	$9^{11}, 8^{16}, 1^{21}$
	5	$15, 11^9, 9^7, 5^{10}, 1^{21}$
	11	$11^{11}, 9^7, 5^8, 3, 1^{21}$
	13	$13^2, 11^8, 9^7, 7, 5^8, 3, 1^{21}$
	others	$15, 11^9, 9^7, 7, 5^8, 3, 1^{21}$
$E_6(a_3)$	2	$8^{20}, 5^{12}, 4^2, 3^2, 1^{14}$
	3	$9^9, 8^2, 6^{16}, 3^{16}, 1^7$
	5	$11, 10^2, 9^7, 7^7, 5^{17}, 3^2, 1^{14}$
	7	$7^{28}, 5^7, 3, 1^{14}$
	others	$11^2, 9^8, 7^8, 5^{15}, 3^3, 1^{14}$
$(D_4 + A_2)^{(2)}$	2	$8^{10}, 7^2, 6^4, 5^4, 4^{24}, 2^6, 1^2$
$D_4 + A_2$	2	$4^{56}, 3^8$
	3	$9^8, 7^7, 6^{12}, 3^{16}, 1^7$
	7	$7^{28}, 5, 3^{13}, 1^8$
	others	$11, 9^6, 7^{14}, 5^7, 3^{14}, 1^8$
$A_4 + A_2 + A_1$	2	$8^{10}, 7^2, 6^4, 5^4, 4^{24}, 2^6, 1^2$
	3	$9^3, 8^2, 7, 6^{18}, 5^3, 3^{25}$
	5	$5^{46}, 4^2, 3^2, 2^2$
	7	$7^{19}, 6^2, 5^7, 4^8, 3^7, 2^6, 1^3$
	others	$9^3, 8^4, 7^5, 6^6, 5^{10}, 4^8, 3^7, 2^6, 1^3$
$D_5(a_1) + A_1$	2	$8^{10}, 7^2, 6^4, 5^4, 4^{24}, 2^6, 1^2$
	3	$9^5, 8^6, 7^4, 6^{12}, 3^{11}, 2^8, 1^6$
	7	$7^{28}, 4^2, 3^6, 2^{10}, 1^6$
	others	$11, 9^3, 8^6, 7^8, 6^6, 5^3, 4^2, 3^7, 2^{10}, 1^6$
A_5	2	$8^8, 7^{12}, 6^{12}, 2^6, 1^{16}$
	3	$9^9, 7^4, 6^{14}, 3^{15}, 1^{10}$
	5	$11, 10^2, 9^7, 6^{14}, 5^9, 4^2, 1^{17}$
	7	$7^{21}, 6^{14}, 1^{17}$
	others	$11, 10^2, 9^7, 7, 6^{14}, 5^7, 4^2, 3, 1^{17}$
$A_4 + A_2$	2	$8^6, 7^{10}, 5^4, 4^{20}, 3^8, 1^6$
	3	$9^3, 7^5, 6^{16}, 5^3, 3^{24}, 1^3$
	5	$5^{46}, 3^5, 1^3$
	7	$7^{19}, 5^{11}, 3^{18}, 1^6$
	others	$9^3, 7^{13}, 5^{14}, 3^{18}, 1^6$
$A_4 + 2A_1$	2	$8^6, 7^2, 6^{12}, 5^4, 4^{14}, 3^2, 2^{16}$
	3	$9, 8^4, 7^3, 6^{12}, 5^7, 4^4, 3^{17}, 2^4, 1^4$
	5	$5^{45}, 3^4, 2^4, 1^3$
	7	$7^{15}, 6^4, 5^8, 4^8, 3^9, 2^8, 1^4$
	others	$9, 8^4, 7^5, 6^8, 5^9, 4^8, 3^9, 2^8, 1^4$
$D_5(a_1)$	2	$8^{10}, 7^2, 5^{12}, 4^{20}, 1^{14}$
	3	$9^3, 8^8, 7^6, 6^{10}, 3^8, 2^8, 1^{15}$
	7	$7^{28}, 3^7, 2^8, 1^{15}$
	others	$11, 9, 8^8, 7^8, 6^8, 5, 3^8, 2^8, 1^{15}$
$2A_3$	2	$4^{56}, 3^8$
	5	$5^{38}, 4^{12}, 1^{10}$
	7	$7^{14}, 5^{10}, 4^{16}, 3^6, 2^4, 1^{10}$
	others	$8^4, 7^6, 6^4, 5^{10}, 4^{16}, 3^6, 2^4, 1^{10}$
$A_4 + A_1$	2	$8^4, 7^6, 6^8, 5^8, 4^{10}, 3^6, 2^{10}, 1^8$
	3	$9, 8^2, 7^7, 6^8, 5^9, 4^6, 3^{12}, 2^6, 1^9$
	5	$5^{45}, 3, 2^6, 1^8$
	7	$7^{13}, 6^6, 5^8, 4^8, 3^8, 2^8, 1^9$
	others	$9, 8^2, 7^7, 6^8, 5^9, 4^8, 3^8, 2^8, 1^9$
$D_4(a_1) + A_2$	2	$4^{56}, 3^8$
	3	$7, 6^{14}, 5^6, 3^{42}, 1$
	5	$5^{36}, 3^{20}, 1^8$
	others	$7^8, 5^{20}, 3^{28}, 1^8$
$D_4 + A_1$	2	$4^{54}, 3^2, 2^{10}, 1^6$
	3	$9^2, 8^6, 7^{13}, 6^6, 3^2, 2^{14}, 1^{21}$
	7	$7^{28}, 3, 2^{14}, 1^{21}$
	others	$11, 8^6, 7^{14}, 6^6, 3^2, 2^{14}, 1^{21}$
$A_3 + A_2 + A_1$	2	$4^{54}, 3^2, 2^{10}, 1^6$
	3	$7, 6^{14}, 5^3, 4^6, 3^{37}, 2^2, 1^3$
	5	$5^{32}, 4^8, 3^{10}, 2^{10}, 1^6$
	others	$7^5, 6^6, 5^{10}, 4^{14}, 3^{15}, 2^{10}, 1^6$
A_4	2	$8^2, 7^{10}, 5^{20}, 4^2, 3^{10}, 1^{24}$
	5	$5^{45}, 1^{23}$
	7	$7^{13}, 5^{20}, 3^{11}, 1^{24}$
	others	$9, 7^{11}, 5^{21}, 3^{11}, 1^{24}$
$(A_3 + A_2)^{(2)}$	2	$4^{54}, 3^2, 2^{10}, 1^6$
$A_3 + A_2$	2	$4^{46}, 3^{18}, 1^{10}$
	3	$7, 6^{12}, 5^5, 4^8, 3^{34}, 1^{10}$
	5	$5^{30}, 4^8, 3^{13}, 2^8, 1^{11}$
	others	$7^3, 6^8, 5^8, 4^{16}, 3^{16}, 2^8, 1^{11}$
$D_4(a_1) + A_1$	2	$4^{54}, 3^2, 2^{10}, 1^6$
	3	$7, 6^8, 5^{12}, 4^6, 3^{28}, 2^8, 1^9$
	5	$5^{29}, 4^6, 3^{14}, 2^{14}, 1^9$
	others	$7^2, 6^6, 5^{13}, 4^{12}, 3^{16}, 2^{14}, 1^9$
$A_3 + 2A_1$	2	$4^{44}, 3^4, 2^{28}, 1^4$
	3	$7, 6^6, 5^{11}, 4^{14}, 3^{19}, 2^{12}, 1^{13}$
	5	$5^{25}, 4^{10}, 3^{14}, 2^{14}, 1^{13}$
	others	$7, 6^6, 5^{11}, 4^{16}, 3^{15}, 2^{14}, 1^{13}$
$2A_2 + 2A_1$	2	$4^{44}, 3^4, 2^{28}, 1^4$
	3	$3^{80}, 2^4$
	5	$5^{18}, 4^{12}, 3^{20}, 2^{20}, 1^{10}$
	others	$6^4, 5^{10}, 4^{16}, 3^{20}, 2^{20}, 1^{10}$
D_4	2	$4^{54}, 3^2, 1^{26}$
	3	$9^2, 7^{25}, 3, 1^{52}$
	7	$7^{28}, 1^{52}$
	others	$11, 7^{26}, 3, 1^{52}$
$D_4(a_1)$	2	$4^{54}, 3^2, 1^{26}$
	3	$7, 6^2, 5^{24}, 3^{27}, 1^{28}$
	5	$5^{29}, 3^{25}, 1^{28}$
	others	$7^2, 5^{25}, 3^{27}, 1^{28}$
$A_3 + A_1$	2	$4^{40}, 3^{12}, 2^{18}, 1^{16}$
	5	$5^{21}, 4^{16}, 3^9, 2^{14}, 1^{24}$
	others	$7, 6^2, 5^{15}, 4^{18}, 3^{10}, 2^{14}, 1^{24}$
$2A_2 + A_1$	2	$4^{40}, 3^{12}, 2^{18}, 1^{16}$
	3	$3^{79}, 2^2, 1^7$
	5	$5^{14}, 4^{14}, 3^{23}, 2^{18}, 1^{17}$
	others	$6^2, 5^{10}, 4^{16}, 3^{23}, 2^{18}, 1^{17}$
$2A_2$	2	$4^{28}, 3^{36}, 1^{28}$
	3	$3^{78}, 1^{14}$
	others	$5^{14}, 3^{50}, 1^{28}$
$A_2 + 3A_1$	2	$4^{26}, 3^6, 2^{62}, 1^2$
	3	$3^{70}, 2^{14}, 1^{10}$
	others	$5^7, 4^{14}, 3^{28}, 2^{28}, 1^{17}$

TABLE 12. Jordan Blocks of nilpotent elements on the adjoint module for E_8, IV

A_3	2	$4^{36}, 3^{20}, 1^{44}$
	5	$5^{13}, 4^{32}, 1^{55}$
	others	$7, 5^{11}, 4^{32}, 3, 1^{55}$
$A_2 + 2A_1$	2	$4^{22}, 3^{14}, 2^{52}, 1^{14}$
	3	$3^{65}, 2^{16}, 1^{21}$
	others	$5^{3}, 4^{16}, 3^{27}, 2^{32}, 1^{24}$
$A_2 + A_1$	2	$4^{14}, 3^{30}, 2^{34}, 1^{34}$
	3	$3^{58}, 2^{20}, 1^{34}$
	others	$5, 4^{12}, 3^{32}, 2^{32}, 1^{35}$
$4A_1$	2	$2^{120}, 1^{8}$
	3	$3^{44}, 2^{40}, 1^{36}$
	others	$4^{8}, 3^{28}, 2^{48}, 1^{36}$
A_2	2	$4^{2}, 3^{54}, 1^{78}$
	3	$3^{57}, 1^{77}$
	others	$5, 3^{55}, 1^{78}$
$3A_1$	2	$2^{110}, 1^{28}$
	3	$3^{31}, 2^{50}, 1^{55}$
	others	$4^{2}, 3^{27}, 2^{52}, 1^{55}$
$2A_1$	2	$2^{92}, 1^{64}$
	others	$3^{14}, 2^{64}, 1^{78}$
A_1	2	$2^{58}, 1^{132}$
	others	$3, 2^{56}, 1^{133}$

TABLE 13. Jordan Blocks of nilpotent elements on the adjoint module for E_8, V

e	p	Jordan Blocks
E_7	2	$16^6, 15^2, 2, 1^5$
	3	$27^3, 9^4, 8^2$
	5	$25^4, 15^2, 3$
	7	$35, 21^4, 7^2$
	13	$35, 27, 23, 19, 13^2, 3$
	17	$35, 27, 23, 17^2, 11, 3$
	19	19^7
	23	$23^5, 15, 3$
	29	$29^2, 27, 19, 15, 11, 3$
	31	$31^2, 23, 19, 15, 11, 3$
	others	$35, 27, 23, 19, 15, 11, 3$
$E_7(a_1)$	2	$16^6, 15^2, 2, 1^5$
	3	$27, 19, 17^3, 9^4$
	5	$25^2, 19, 17, 15, 11^2, 7, 3$
	7	$27, 21^2, 15, 14^2, 7^3$
	11	$23, 22^2, 11^6$
	13	$27, 23, 19, 17, 13^2, 11, 7, 3$
	17	$17^7, 11, 3$
	19	$19^5, 17, 11, 7, 3$
	23	$23^3, 17, 15, 11^2, 7, 3$
	others	$27, 23, 19, 17, 15, 11^2, 7, 3$
$E_7(a_2)$	2	$16^4, 13^4, 4^2, 3^2, 2, 1$
	3	$19, 17^3, 9^7$
	5	$23, 19, 17, 15^2, 11, 10^2, 7, 3^2$
	13	$13^{10}, 3$
	17	$17^5, 15, 11, 9, 7, 3^2$
	19	$19^3, 17, 15, 11^2, 9, 7, 3^2$
	others	$23, 19, 17, 15^2, 11^2, 9, 7, 3^2$
$E_7(a_3)$	2	$16^2, 15^2, 9^4, 8^4, 2, 1$
	3	$19, 17, 15^2, 11^2, 9^3, 6^2, 3^2$
	5	$19, 17, 15^2, 11^2, 10^2, 7, 5^3, 3$
	11	$11^{11}, 9, 3$
	13	$13^8, 11, 7, 5, 3^2$
	17	$17^3, 15, 11^3, 9, 7^2, 5, 3^2$
	others	$19, 17, 15^2, 11^3, 9, 7^2, 5, 3^2$
E_6	2	$16^4, 13^4, 4^2, 3^2, 1^3$
	3	$9^{13}, 8^2$
	13	$13^{10}, 1^3$
	17	$17^5, 15, 9^3, 3, 1^3$
	19	$19^2, 17^3, 11, 9^3, 3, 1^3$
	others	$23, 17^3, 15, 11, 9^3, 3, 1^3$
$E_6(a_1)$	2	$16^2, 13^2, 11^2, 9^2, 8^2, 5^2, 4^2, 1$
	3	$9^{13}, 8^2$
	5	$17, 15, 13^2, 11, 10^2, 9^2, 7, 5^3, 3, 1$
	7	$15, 14^2, 13^2, 7^9, 1$
	11	$11^{10}, 9, 5^2, 3, 1$
	13	$13^6, 11, 9^2, 7, 5^3, 3, 1$
	others	$17, 15, 13^2, 11^2, 9^3, 7, 5^3, 3, 1$
D_6	2	$8^{14}, 7^2, 2, 1^5$
	3	$19, 16^2, 15, 11, 10^2, 9^2, 6^2, 3, 1^3$
	5	$19, 16^2, 15, 11^2, 10^2, 6^2, 5^2, 1^3$
	11	$11^{10}, 10^2, 1^3$
	13	$13^8, 11, 6^2, 3, 1^3$
	17	$17^2, 16^2, 11^2, 10^2, 7, 6^2, 3, 1^3$
	others	$19, 16^2, 15, 11^2, 10^2, 7, 6^2, 3, 1^3$
$E_7(a_4)$	2	$8^{14}, 7^2, 2, 1^5$
	3	$9^{11}, 8^2, 6^2, 3^2$
	5	$15, 13, 11^2, 10^4, 7, 5^6, 3^2$
	11	$11^8, 9, 7^2, 5^2, 3^4$
	13	$13^3, 11^3, 9^2, 7^3, 5^2, 3^4$
	others	$15, 13, 11^4, 9^2, 7^3, 5^2, 3^4$

e	p	Jordan Blocks
$D_6(a_1)$	2	$8^{14}, 7^2, 2, 1^5$
	3	$9^{11}, 7^4, 3, 1^3$
	5	$15, 12^2, 11, 10^4, 7, 5^6, 3, 1^3$
	11	$11^8, 9, 7, 6^2, 4^2, 3^2, 1^3$
	13	$13^2, 12^2, 11, 10^2, 9, 7^2, 6^2, 4^2, 3^2, 1^3$
	others	$15, 12^2, 11^2, 10^2, 9, 7^2, 6^2, 4^2, 3^2, 1^3$
D_5+A_1	2	$8^{12}, 5^4, 4^2, 3^2, 2, 1$
	3	$9^{11}, 7^4, 3, 1^3$
	5	$15, 12^2, 11, 10^2, 9^3, 5^6, 3, 1^3$
	11	$11^7, 9^3, 6^2, 4^2, 3^2, 1^3$
	13	$13^2, 12^2, 10^2, 9^3, 7, 6^2, 4^2, 3^2, 1^3$
	others	$15, 12^2, 11, 10^2, 9^3, 7, 6^2, 4^2, 3^2, 1^3$
$(A_6)^{(2)}$	2	$8^{14}, 7^2, 2, 1^5$
A_6	2	$8^{14}, 7^2, 1^7$
	3	$9^{11}, 7^4, 3, 1^3$
	7	7^{19}
	11	$11^7, 7^5, 5^3, 3, 1^3$
	others	$13^3, 11, 9^3, 7^5, 5^3, 3, 1^3$
$E_7(a_5)$	2	$8^{12}, 5^4, 4^2, 3^2, 2, 1$
	3	$9^9, 7, 6^2, 5^3, 3^6$
	5	$11, 10^4, 9, 7^2, 5^{10}, 3^3$
	7	$7^{17}, 5, 3^3$
	others	$11^3, 9^3, 7^5, 5^4, 3^6$
D_5	2	$8^{12}, 5^4, 4^2, 3^2, 1^3$
	3	$9^7, 8^8, 1^6$
	5	$15, 11^5, 9^3, 5^6, 1^6$
	11	$11^7, 9^3, 5^4, 3, 1^6$
	13	$13^2, 11^4, 9^3, 7, 5^4, 3, 1^6$
	others	$15, 11^5, 9^3, 7, 5^4, 3, 1^6$
$E_6(a_3)$	2	$8^{12}, 5^4, 4^2, 3^2, 1^3$
	3	$9^5, 8^2, 6^8, 3^8$
	5	$11, 10^2, 9^3, 7^3, 5^9, 3^2, 1^3$
	7	$7^{16}, 5^3, 3, 1^3$
	others	$11^2, 9^4, 7^4, 5^7, 3^3, 1^3$
$D_6(a_2)$	2	$8^8, 7^4, 6^4, 2^7, 1^3$
	3	$9^9, 7, 6^2, 5, 4^4, 3^3, 1^3$
	5	$11, 10^4, 8^2, 7, 5^9, 4^2, 3, 1^3$
	7	$7^{17}, 4^2, 3, 1^3$
	others	$11^2, 10^2, 9, 8^2, 7^3, 6^2, 5, 4^4, 3^3, 1^3$
$D_5(a_1)+A_1$	2	$8^6, 7^2, 5^4, 4^{12}, 2, 1$
	3	$9^5, 7^4, 6^6, 3^7, 1^3$
	7	$7^{16}, 3^6, 1^3$
	others	$11, 9^3, 7^8, 5^3, 3^7, 1^3$
A_5+A_1	2	$8^8, 7^4, 6^4, 2^7, 1^3$
	3	$9^9, 3^{16}, 2^2$
	5	$11, 10^2, 9^3, 8^2, 5^9, 3, 2^4, 1^3$
	7	$7^{17}, 3, 2^4, 1^3$
	others	$11, 10^2, 9^3, 8^2, 7, 6^2, 5^3, 4^2, 3^2, 2^4, 1^3$
$(A_5)'$	2	$8^8, 7^4, 6^4, 2^6, 1^5$
	3	$9^5, 7^4, 6^6, 3^7, 1^3$
	5	$11, 10^2, 9^3, 6^6, 5^5, 4^2, 1^6$
	7	$7^{13}, 6^6, 1^6$
	others	$11, 10^2, 9^3, 7, 6^6, 5^3, 4^2, 3, 1^6$
A_4+A_2	2	$8^6, 7^2, 5^4, 4^{12}, 1^3$
	3	$9^3, 7, 6^8, 5^3, 3^{12}$
	5	$5^{26}, 3$
	7	$7^{11}, 5^7, 3^6, 1^3$
	others	$9^3, 7^5, 5^{10}, 3^6, 1^3$
$D_5(a_1)$	2	$8^6, 7^2, 5^4, 4^{12}, 1^3$
	3	$9^3, 8^4, 7^2, 6^6, 3^4, 2^4, 1^4$
	7	$7^{16}, 3^3, 2^4, 1^4$
	others	$11, 9, 8^4, 7^4, 6^4, 5, 3^4, 2^4, 1^4$

TABLE 14. Jordan Blocks of nilpotent elements on the adjoint module for E_7, I

e	p	Jordan Blocks
A_4+A_1	2	$8^4, 7^2, 6^4, 5^4, 4^6, 3^2, 2^6, 1$
	3	$9, 8^2, 7^3, 6^4, 5^5, 4^2, 3^8, 2^2, 1^2$
	5	$5^{25}, 3, 2^2, 1$
	7	$7^9, 6^2, 5^4, 4^4, 3^4, 2^4, 1^2$
	others	$9, 8^2, 7^3, 6^4, 5^5, 4^4, 3^4, 2^4, 1^2$
D_4+A_1	2	$4^{30}, 3^2, 2, 1^5$
	3	$9^2, 8^4, 7^5, 6^4, 3^2, 2^4, 1^{10}$
	7	$7^{16}, 3, 2^4, 1^{10}$
	others	$11, 8^4, 7^6, 6^4, 3^2, 2^4, 1^{10}$
$(A_5)''$	2	$8^4, 7^{12}, 2, 1^{15}$
	3	$9^9, 3^{15}, 1^7$
	5	$11, 9^7, 5^9, 1^{14}$
	7	$7^{17}, 1^{14}$
	others	$11, 9^7, 7, 5^7, 3, 1^{14}$
$A_3+A_2+A_1$	2	$4^{30}, 3^2, 2, 1^5$
	3	$7, 6^8, 5^3, 3^{21}$
	5	$5^{20}, 3^{10}, 1^3$
	others	$7^5, 5^{10}, 3^{15}, 1^3$
A_4	2	$8^2, 7^6, 5^8, 4^2, 3^6, 1^9$
	5	$5^{25}, 1^8$
	7	$7^9, 5^8, 3^7, 1^9$
	others	$9, 7^7, 5^9, 3^7, 1^9$
$(A_3+A_2)^{(2)}$	2	$4^{30}, 3^2, 2, 1^5$
A_3+A_2	2	$4^{30}, 3^2, 1^7$
	3	$7, 6^8, 5, 4^4, 3^{18}, 1^3$
	5	$5^{18}, 4^4, 3^5, 2^4, 1^4$
	others	$7^3, 6^4, 5^4, 4^8, 3^8, 2^4, 1^4$
$D_4(a_1)+A_1$	2	$4^{30}, 3^2, 2, 1^5$
	3	$7, 6^6, 5^4, 4^4, 3^{16}, 1^6$
	5	$5^{17}, 4^4, 3^6, 2^4, 1^6$
	others	$7^2, 6^4, 5^5, 4^8, 3^8, 2^4, 1^6$
D_4	2	$4^{30}, 3^2, 1^7$
	3	$9^2, 7^{13}, 3, 1^{21}$
	7	$7^{16}, 1^{21}$
	others	$11, 7^{14}, 3, 1^{21}$
A_3+2A_1	2	$4^{24}, 3^4, 2^{11}, 1^3$
	3	$7, 6^4, 5^7, 4^4, 3^{11}, 2^6, 1^6$
	5	$5^{17}, 4^2, 3^6, 2^8, 1^6$
	others	$7, 6^4, 5^7, 4^6, 3^7, 2^8, 1^6$
$D_4(a_1)$	2	$4^{30}, 3^2, 1^7$
	3	$7, 6^2, 5^{12}, 3^{15}, 1^9$
	5	$5^{17}, 3^{13}, 1^9$
	others	$7^2, 5^{13}, 3^{15}, 1^9$

e	p	Jordan Blocks
$(A_3+A_1)'$	2	$4^{24}, 3^4, 2^{10}, 1^5$
	5	$5^{13}, 4^8, 3^5, 2^6, 1^9$
	others	$7, 6^2, 5^7, 4^{10}, 3^6, 2^6, 1^9$
$2A_2+A_1$	2	$4^{24}, 3^4, 2^{10}, 1^5$
	3	$3^{43}, 2^2$
	5	$5^{10}, 4^6, 3^{11}, 2^{10}, 1^6$
	others	$6^2, 5^6, 4^8, 3^{11}, 2^{10}, 1^6$
$(A_3+A_1)''$	2	$4^{20}, 3^{12}, 2, 1^{15}$
	5	$5^{17}, 3^9, 1^{21}$
	others	$7, 5^{15}, 3^{10}, 1^{21}$
A_2+3A_1	2	$4^{14}, 3^6, 2^{29}, 1$
	3	$3^{42}, 1^7$
	others	$5^7, 3^{28}, 1^{14}$
$2A_2$	2	$4^{20}, 3^{12}, 1^{17}$
	3	$3^{42}, 1^7$
	others	$5^{10}, 3^{22}, 1^{17}$
A_3	2	$4^{20}, 3^{12}, 1^{17}$
	5	$5^9, 4^{16}, 1^{24}$
	others	$7, 5^7, 4^{16}, 3, 1^{24}$
A_2+2A_1	2	$4^{14}, 3^6, 2^{28}, 1^3$
	3	$3^{37}, 2^8, 1^6$
	others	$5^3, 4^8, 3^{15}, 2^{16}, 1^9$
A_2+A_1	2	$4^{10}, 3^{14}, 2^{18}, 1^{15}$
	3	$3^{34}, 2^8, 1^{15}$
	others	$5, 4^8, 3^{16}, 2^{16}, 1^{16}$
$4A_1$	2	$2^{63}, 1^7$
	3	$3^{28}, 2^{14}, 1^{21}$
	others	$4^6, 3^{16}, 2^{20}, 1^{21}$
A_2	2	$4^2, 3^{30}, 1^{35}$
	3	$3^{33}, 1^{34}$
	others	$5, 3^{31}, 1^{35}$
$(3A_1)'$	2	$2^{62}, 1^9$
	3	$3^{19}, 2^{26}, 1^{24}$
	others	$4^2, 3^{15}, 2^{28}, 1^{24}$
$(3A_1)''$	2	$2^{53}, 1^{27}$
	others	$3^{27}, 1^{52}$
$2A_1$	2	$2^{52}, 1^{29}$
	others	$3^{10}, 2^{32}, 1^{39}$
A_1	2	$2^{34}, 1^{65}$
	others	$3, 2^{32}, 1^{66}$

Table 15. Jordan Blocks of nilpotent elements on the adjoint module for E_7, II

e	p	Jordan Blocks
E_6	2	$16^4, 4^2, 3^2$
	3	$9^6, 8^3$
	13	13^6
	17	$17^3, 15, 9, 3$
	19	$19^2, 17, 11, 9, 3$
	others	$23, 17, 15, 11, 9, 3$
$E_6(a_1)$	2	$16^2, 11^2, 8^2, 4^2$
	3	$9^6, 8^3$
	5	$17, 15, 11, 10^2, 7, 5, 3$
	7	$15, 14^2, 7^5$
	11	$11^6, 9, 3$
	13	$13^4, 11, 7, 5, 3$
	others	$17, 15, 11^2, 9, 7, 5, 3$
D_5	2	$8^8, 4^2, 3^2$
	3	$9^5, 8^4, 1$
	5	$15, 11^3, 9, 5^4, 1$
	11	$11^5, 9, 5^2, 3, 1$
	13	$13^2, 11^2, 9, 7, 5^2, 3, 1$
	others	$15, 11^3, 9, 7, 5^2, 3, 1$
$E_6(a_3)$	2	$8^8, 4^2, 3^2$
	3	$9^3, 8^2, 6^4, 3^3, 2$
	5	$11, 10^2, 9, 7, 5^5, 3^2$
	7	$7^{10}, 5, 3$
	others	$11^2, 9^2, 7^2, 5^3, 3^3$
$D_5(a_1)$	2	$8^4, 7^2, 4^8$
	3	$9^3, 8^2, 6^4, 3^2, 2^2, 1$
	7	$7^{10}, 3, 2^2, 1$
	others	$11, 9, 8^2, 7^2, 6^2, 5, 3^2, 2^2, 1$
A_5	2	$8^8, 2^6, 1^2$
	3	$9^3, 7^4, 6^2, 3^2, 2, 1^3$
	5	$11, 10^2, 9, 6^2, 5^3, 4^2, 1^3$
	7	$7^9, 6^2, 1^3$
	others	$11, 10^2, 9, 7, 6^2, 5, 4^2, 3, 1^3$
$A_4 + A_1$	2	$8^4, 6^2, 5^2, 4^4, 2^4$
	3	$9, 8^2, 7, 6^2, 5^3, 3^6, 1$
	5	$5^{15}, 3$
	7	$7^7, 5^2, 4^2, 3^2, 2^2, 1$
	others	$9, 8^2, 7, 6^2, 5^3, 4^2, 3^2, 2^2, 1$

e	p	Jordan Blocks
D_4	2	$4^{18}, 3^2$
	3	$9^2, 7^7, 3, 1^8$
	7	$7^{10}, 1^8$
	others	$11, 7^8, 3, 1^8$
A_4	2	$8^2, 7^4, 5^2, 4^2, 3^4, 1^4$
	5	$5^{15}, 1^3$
	7	$7^7, 5^2, 3^5, 1^4$
	others	$9, 7^5, 5^3, 3^5, 1^4$
$D_4(a_1)$	2	$4^{18}, 3^2$
	3	$7, 6^2, 5^6, 3^9, 1^2$
	5	$5^{11}, 3^7, 1^2$
	others	$7^2, 5^7, 3^9, 1^2$
$A_3 + A_1$	2	$4^{16}, 2^6, 1^2$
	5	$5^9, 4^4, 3^3, 2^2, 1^4$
	others	$7, 6^2, 5^3, 4^6, 3^4, 2^2, 1^4$
$2A_2 + A_1$	2	$4^{16}, 2^6, 1^2$
	3	$3^{24}, 2^3$
	5	$5^8, 4^2, 3^5, 2^6, 1^3$
	others	$6^2, 5^4, 4^4, 3^5, 2^6, 1^3$
A_3	2	$4^{12}, 3^8, 1^6$
	5	$5^7, 4^8, 1^{11}$
	others	$7, 5^5, 4^8, 3, 1^{11}$
$A_2 + 2A_1$	2	$4^{10}, 3^2, 2^{16}$
	3	$3^{23}, 2^4, 1$
	others	$5^3, 4^4, 3^9, 2^8, 1^4$
$2A_2$	2	$4^{16}, 1^{14}$
	3	$3^{23}, 2, 1^7$
	others	$5^8, 3^8, 1^{14}$
$A_2 + A_1$	2	$4^8, 3^6, 2^{10}, 1^8$
	3	$3^{22}, 2^2, 1^8$
	others	$5, 4^6, 3^8, 2^8, 1^9$
A_2	2	$4^2, 3^{18}, 1^{16}$
	3	$3^{21}, 1^{15}$
	others	$5, 3^{19}, 1^{16}$
$3A_1$	2	$2^{38}, 1^2$
	3	$3^{13}, 2^{14}, 1^{11}$
	others	$4^2, 3^9, 2^{16}, 1^{11}$
$2A_1$	2	$2^{32}, 1^{14}$
	others	$3^8, 2^{16}, 1^{22}$
A_1	2	$2^{22}, 1^{34}$
	others	$3, 2^{20}, 1^{35}$

TABLE 16. Jordan Blocks of nilpotent elements on the adjoint module for E_6

e	p	Jordan Blocks
F_4	2	$16^2, 4^2, 3^4$
	3	$9^4, 8^2$
	13	13^4
	17	$17^2, 15, 3$
	19	$19^2, 11, 3$
	others	$23, 15, 11, 3$
$F_4(a_1)$	2	$8^4, 4^2, 3^4$
	3	$9^4, 8^2$
	5	$15, 11^2, 5^3$
	11	$11^4, 5, 3$
	13	$13^2, 11, 7, 5, 3$
	others	$15, 11^2, 7, 5, 3$
$F_4(a_2)$	2	$8^4, 4^2, 3^4$
	3	$9^2, 8^2, 6^2, 3^2$
	5	$11, 10^2, 5^3, 3^2$
	7	$7^7, 3$
	others	$11^2, 9, 7, 5, 3^3$
$(C_3)^{(2)}$	2	$8^4, 4, 3^3, 2^3, 1$
C_3	2	$8^4, 2^8, 1^4$
	3	$9^2, 7^4, 3, 1^3$
	5	$11, 10^2, 5^2, 4^2, 1^3$
	7	$7^7, 1^3$
	others	$11, 10^2, 7, 4^2, 3, 1^3$
B_3	2	$4^{10}, 3^4$
	3	$9^2, 7^4, 3, 1^3$
	7	$7^7, 1^3$
	others	$11, 7^5, 3, 1^3$
$F_4(a_3)$	2	$4^{10}, 3^4$
	3	$7, 6^2, 5^3, 3^6$
	5	$5^8, 3^4$
	others	$7^2, 5^4, 3^6$
$C_3(a_1)^{(2)}$	2	$4^9, 3^3, 2^3, 1$
$C_3(a_1)$	2	$4^8, 2^8, 1^4$
	5	$5^7, 4^2, 3^2, 1^3$
	others	$7, 6^2, 5, 4^4, 3^3, 1^3$
$(\tilde{A}_2 + A_1)^{(2)}$	2	$4^9, 3^3, 2^3, 1$
$\tilde{A}_2 + A_1$	2	$4^8, 2^8, 1^4$
	3	$3^{16}, 2^2$
	5	$5^7, 3^2, 2^4, 1^3$
	others	$6^2, 5^3, 4^2, 3^2, 2^4, 1^3$
$(B_2)^{(2)}$	2	$4^7, 3^7, 1^3$
B_2	2	$4^6, 3^4, 2^5, 1^6$
	5	$5^6, 4^4, 1^6$
	others	$7, 5^4, 4^4, 3, 1^6$
$A_2 + \tilde{A}_1$	2	$4^6, 3^4, 2^8$
	3	$3^{16}, 2^2$
	others	$5^3, 4^2, 3^6, 2^4, 1^3$
$\tilde{A}_2$	2	$4^8, 1^{20}$
	3	$3^{15}, 1^7$
	others	$5^7, 3, 1^{14}$
$(A_2)^{(2)}$	2	$4^2, 3^{12}, 1^8$
A_2	2	$4^2, 3^{12}, 1^8$
	3	$3^{15}, 1^7$
	others	$5, 3^{13}, 1^8$
$A_1 + \tilde{A}_1$	2	$2^{24}, 1^4$
	3	$3^{10}, 2^8, 1^6$
	others	$4^2, 3^6, 2^{10}, 1^6$
$(\tilde{A}_1)^{(2)}$	2	$2^{21}, 1^{10}$
$\tilde{A}_1$	2	$2^{16}, 1^{20}$
	others	$3^7, 2^8, 1^{15}$
A_1	2	$2^{16}, 1^{20}$
	others	$3, 2^{14}, 1^{21}$

e	p	Jordan Blocks
G_2	2	$4^2, 3^2$
	2	$4^2, 3^2$
	3	$9, 3, 2$
	7	7^2
	others	$11, 3$
$G_2(a_1)$	2	$4^2, 3^2$
	2	$4^2, 3^2$
	3	$3^4, 2$
	others	$5, 3^3$
$(\tilde{A}_1)^{(3)}$	3	$3^3, 2^2, 1$
$\tilde{A}_1$	2	$2^6, 1^2$
	2	$2^6, 1^2$
	3	$3^3, 1^5$
	others	$4^2, 3, 1^3$
A_1	2	$2^6, 1^2$
	2	$2^6, 1^2$
	3	$3, 2^4, 1^3$
	others	$4^2, 3, 1^3$

TABLE 17. Jordan Blocks of nilpotent elements on the adjoint modules for F_4 and G_2

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